Week 1 (II)

Ordered Rings and Fields

($*$) We assume that the rings we consider in this section have nonzero unity 1.

以前的概念：$a < b$ 若且唯若 $b - a$ 是正數, 實數就是如此定義而複數就有困難。

現在該有的概念：先判斷在環中, 那些是"正"的元素。

Defn. An ordered ring is a ring R together with a nonempty set P of R such that

(1) closure $\forall\ a, b \in P$, both $a + b$ and $a \cdot b$ are in P.

(2) Trichotomy $\forall\ a \in R$, exactly one of the following three statements holds, $a \in P$, $a = 0$, $-a \in P$.

Elenents of P are called "positive".

Property 1 If a is a nonzero element of an ordered ring R with positive set P, then $a^2 \in P$. ($\mathbb{C}$ can not be ordered, since $1 = 1^2$ and $i^2 = -1$.)

Proof. Either $a \in P$ or $-a \in P$. If $a \in P$, then $a \cdot a = a^2 \in P$. On the other hand, if $-a \in P$, then $(-a)(-a) = a^2 \in P$. ■

Property 2 $1 \in P$.

Property 3 An ordered ring R is of characteristic 0, i.e., ch(R)=0.

Proof. Since $1 \in P$, $1 + 1 \in P$, $\cdots$, $1 + 1 + \cdots + 1 \in P$, never be zero. ■

Property 4 $0 \notin P$.

Proof. Since $-0 = 0$, $0 \notin P$. (違反 Trichotomy)

Property 5 Every ordered ring contains $\mathbb{Z}$ as a ordered subring.

Proof. $1, 1+1, 1+1+1, \cdots$ are distinct.

Property 6 Every ordered ring R contains no zero divisors.

Proof. Let a and b be two nonzero elements in R. Then a or $-a$ is in P, b or $-b$ is in P. This implies ab or $-ab$ in P but not both. Hence $ab \neq 0$. ■

Defn. $a < b$ if and only if $b - a \in P$.

Property 7

Trichotomy	(a)	Exactly one of the followings holds : $a < b$, $a = b$, $b < a$.
Transitivity	(b)	If $a < b$, $b < c$, then $a < c$.
Isotonicity	(c)	If $b < c$, then $a + b < a + c$.
		If $b < c$, $0 < a$, then $ab < ac$ and $ba < ca$

Proof.

(a) Let $a, b \in R$. Consider $b - a \in R$. Then exactly one of $b - a \in P$, $b - a = 0$, $-(b - a) = a - b \in P$ holds. By def. one of $a < b$, $a = b$, $b < a$ holds.

(b) $(b - a) \in P$, $(c - b) \in P \Rightarrow (b - a) + (c - b) \in P$, as $a < c$.

(c) 自己試試.

Defn. (Archimedean Ring)

An ordered ring R is an Archimedian ring if the following property holds : $\forall\ a, b \in P \subseteq R$, $\exists$ a positive integer n s.t. $na > b$.

More examples of ordered rings

1. Let R be an ordered ring with set P of positive elements. Consider $R[x]$. Let P_{low}=$\{f(x) \mid$ the coefficient of the "lowest power" term of $f(x)$ is in $P\}$ and P_{high}=$\{f(x) \mid$ the coefficient of the "highest power" term of $f(x)$ is in $P\}$.
   For example, in $\mathbb{Z}[x]$, using P_{low}, $-2x + 3x^4$ is not positive. But, using P_{high}, $-2x + 3x^4$ is positive.

2. Using P_{low} and considering $R[x]$ where $\mathbb{R} = R$, we have $(a \in R, a > 0)$ $a - x \in P_{low}$, hence $x < a$. Similary, $x^2 < x$, $0 < \cdots < x^3 < x^2 < x < a$. Therefore, there are infinitely many positive elements of P_{low} which are less than a positive number in $\mathbb{R}$.

3. The nature ordered ring $\mathbb{R}$ itself is an Archimedian ordering, but the ordered ring $\mathbb{R}[x]$ with positive elements defined by P_{low} is not an Archimedian ring.
   Sol. Consider $7(\leftarrow b)$ and $x(\leftarrow a)$, $nx < 7 \quad \forall\ n$.

4. **Formal Power Series Rings**
   Formal $\Rightarrow$ 因爲不討論級數是否收斂, 通常加入"Formal"來表示概念。

Defn. A formal power series in x with coefficients $a_i \in R$ is written as $\Sigma_{i=0}^{\infty} a_i x^i$. (多項式要加入：除了有限個"$i$"之外, 其它的$a_i = 0$.) 所有上述 formal power series in x with coeff. $a_i \in R$ 所成的集合以 $R[[x]]$ 表示.

Property 8 $R[[x]]$ contains $R[x]$ as a subring.

Property 9 Using P_{low} for the positive elements in $R[[x]]$, we have an ordered ring.

(Note) We can not use P_{high}. Some formal power series can not be juged whether it is positive or not.

R is a field with ordering $\Rightarrow$ R is an ordered field.

Example $\mathbb{R}$ is an ordered field.

Example Formal Laurent series with coef. in a field F, $F((x))$, is a field. By using P_{low}, $F((x))$ is an ordered field. (A series $\sum_{i=-\infty}^{\infty} a_i x^i$ where all but a finite number of the a_i are zero for negative values i is called a formal Laurent series.))

Property 10 $F((x))$ contains a subfield : a field of quotients of $F[x]$ and thus **induces** an ordering on this field of quotients.

Defn. If R is an ordered ring with positive set P and $S \leq R$ is a subring of R, then $P \cap S (\neq \phi)$ is a positive set in S and thus induces an ordering of S.

另一種形式的 ”Induce” 來自環同構 (Ring isomorphism).

Theorem Let R be an ordered ring with positive set P and let $\varphi : R \to R'$ be a ring isomorphism. Then R' is an ordered ring with positive set $\varphi(P) = P'$.

Proof. $\forall\ a', b' \in P'$, $a' + b' = \varphi(a) + \varphi(b) = \varphi(a + b)$ where $a,\ b \in P$. Hence $a' + b' \in P'$. similary $a'b' = \varphi(a)\varphi(b) = \varphi(ab) \in P'$. ■

Theorem Let D be an ordered integral domain with positive set P, and let F be a field of quotients of D. The set
P'=$\{x \in F \mid x = \frac{a}{b}$ for $a, b \in D$ and $ab \in P\}$ gives an order on F that induces the given order on D (i.e. $D \cap P'$ is positive in D).
Furthermore, P' is the only subset of F with this property.

proof. 參考課本 p.232-233.

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我只看我有的, 不看我沒有的。

(∗∗) Order 決定於 Positive set.