Algebra (I), Test 1.

10,19

1. Prove or disprove the following statements. (40 points)
   (a) $\langle \mathbb{Z}_7, + \rangle$ is a group.
   (b) Every group with 6 elements is abelian.
   (c) Let $S = \mathbb{R} \setminus \{1\}$ and $a * b = a + b + ab$ for all $a, b \in S$. Then $\langle S, * \rangle$ is a group.
   (d) Any two groups of order 4 are isomorphic.

2. Prove that a nonempty set G, together with an associate binary operation $*$ on G such that $a * x = b$ and $y * a = b$ have solutions in G for all $a, b \in G$, is a group. (20 points)

3. Let H be a subgroup of G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation. (10 points)

4. Prove that a subgroup of a cyclic group is also cyclic. (15 points)

5. Let $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\}$. Prove that $\langle \mathbb{Z}_n^*, \cdot \rangle$ is a group if and only if n is a prime. (15 points)

6. Let $\phi : G \to G'$ be an isomorphism of groups G and G'. Prove that $\phi[G] = \{\phi(x) | x \in G\}$ is a subgroup of G'. (15 points)