Definition (Prefix transposition)

A prefix transposition is a transposition $\tau(i,j,k)$ defined earlier with $i=1$. prefix transposition distance of π : $ptd(\pi)$

e.g. $\rho = \begin{pmatrix} 1 & 2 & \cdots & \overset{i-1}{i} & i+1 & \cdots & j-1 & j & j+1 & \cdots & k-1 & k & \cdots & n \\ \boxed{j \ j+1 \ \cdots \ k-1} & \boxed{1 \ 2 \ \cdots \ j-1} & k & \cdots & n \end{pmatrix}$ $\pi = (\pi_1 \, \pi_2 \, \cdots \, \pi_n)$

$\pi \circ \rho = (\pi_j \, \pi_{j+1} \cdots \pi_{k-1} \boxed{\pi_1 \pi_2 \cdots \pi_{j-1}} \pi_k \cdots \pi_n)$

(•) 把最前面的一段移到後面放在 π_{k-1} 与 π_k 之间。
($j-1$ 項)

Definition (Prefix transposition breakpoints)

A prefix transposition breakpoint in a permutation $\pi \in S_n$ is a breakpoint of π, except $(0, \pi_1)$ is always a prefix transposition breakpoint, this number is denoted by $ptb(\pi)$.

(•) A k-prefix transposition is a transposition τ such that

$ptb(\pi \circ \tau) = ptb(\pi) + k$.

(*) $k \in \{-2,-1,0,1,2\} \Rightarrow ptd(\pi) \geq \left\lceil \frac{ptb(\pi)-1}{2} \right\rceil$.

Good Lower bound!

(**) $ptd(\pi) \geq td(\pi)$.

<u>Known results</u> (Upper Bounds)

1. $\forall\ \pi \in S_n$, $ptd(\pi) \le ptb(\pi) - 2$.

   Proof. We can remove at least one breakpoint at each step. ∎

2. $\forall\ \pi \in S_n$, $ptd(\pi) \le n - \log_8 n$.

   Proof. Use your imaginatron! See, "Bounding prefix transposition distance for strings and permutations" by B. Chitturi and H. Sudborough.

3. $\forall\ \pi \in S_n$, $ptd(\pi) \le \left\lfloor \frac{3n+1}{4} \right\rfloor$.

   See "Edit distances and factorizations of even permutations" by A. Labarre, LNCS, 5193 (2008), 635–646.

4. $\forall\ \pi \in S_n$, there exists at most one prefix transposition τ such that $ptb(\pi \circ \tau) = ptb(\pi) - 2$.

In general, determining $ptd(\pi)$ for general permutation π is very difficult. But, for some special permutations, there are good answers.

e.g. $\pi = (k{+}1\ \ k\ \ k{+}2\ \ k{-}1\ \cdots\ 2k{-}1\ \ 2\ \ 2k\ \ 1)$, $ptd(\pi) = k$.
(Can you do it?)

$\pi = (k{+}1\ \ k\ \ k{+}2\ \ k{-}1\ \cdots\ 2k{-}1\ \ 2\ \ 2k\ \ 1)$

Answer for $\mathrm{ptd}(\pi) = k$.

From the following example, you can see the pattern.

$k = 6$.

(7 6 8 5 9 4 10 3 11 2 12 1)

1

(6 7 8 5 9 4 10 3 11 2 12 1).

2

(5 6 7 8 9 4 10 3 11 2 12 1)

3

(4 5 6 7 8 9 10 3 11 2 12 1)

4

(3 4 5 6 7 8 9 10 11 2 12 1

5

(2 3 4 5 6 7 8 9 10 11 12 1)

6

(1 2 3 4 5 6 7 8 9 10 11 12)

$$\tau(1,2,3) \longrightarrow \tau(1,4,5) \longrightarrow \tau(1,6,7) \longrightarrow \cdots$$

$$\longrightarrow \tau(1, 2k, 2k+1)$$

after π_{2k}

Among the study of prefix transpositions, the case $\tau(1,2,k)$ has more impact. It is known as "insertion of leading elements".

Theorem The number of insertions required to sort a permutation $\pi \in S_n$ by head insertions is $n-h$, where h is the largest integer such that the last h elements of π form an increasing substring.

Proof. Let $ptd_1(\pi)$ denote the distance by using transpositions $\tau(1,2,k)$. Since we can insert the leading element to a position which increase the length h of the increasing substring, the upper bound can be obtained subsequently. By the way, there are $n-h$ elements to insert, thus $ptd_1(\pi) \le n-h$.

On the other hand, since the increasing substring starts at π_{n-h+1}, the element π_{n-h} is larger than π_{n-h+1} and thus π_{n-h} has to be moved to somewhere to the right of π_{n-h+1}, so are $\pi_1, \pi_2, \cdots, \pi_{n-h-1}$. Hence, at least $n-h$ insertions are needed. This concludes the proof. ■
$(ptd_1(\pi) \ge n-h)$

e.g. Leading (Insertion)

$$(5\ 4\ 1\ 6\ 3\ 2) \quad h=1$$
$$\Downarrow \tau_1$$
$$(4\ 1\ 6\ 3\ 2\ 5)$$
$$\Downarrow \tau_2$$
$$(1\ 6\ 3\ 2\ 4\ 5)$$
$$\Downarrow \tau_3$$
$$(6\ 3\ 1\ 2\ 4\ 5)$$
$$\Downarrow \tau_4$$
$$(3\ 1\ 2\ 4\ 5\ 6)$$
$$\Downarrow \tau_5$$
$$(1\ 2\ 3\ 4\ 5\ 6)$$

e.g. By reversals

$$(\underline{5\ 4\ 1}\ 6\ 3\ 2)$$
$$\Downarrow$$
$$(1\ \underline{4\ 5\ 6\ 3\ 2})$$
$$\Downarrow$$
$$(1\ 2\ 3\ \underline{6\ 5\ 4})$$
$$\Downarrow$$
$$(1\ 2\ 3\ 4\ 5\ 6)$$

Definition

Reversal $\rho(i,j)$

$$(\pi_1 \pi_2 \cdots \pi_n)$$
$$\Downarrow \rho(i,j)$$
$$(\pi_1 \pi_2 \cdots \pi_{i-1} \pi_j \pi_{j-1} \cdots \pi_i \pi_{j+1} \cdots \pi_n)$$

$$\begin{pmatrix} 1 & 2 & \cdots & i-1 & i & \cdots & j & j+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & j & \cdots & i & j+1 & \cdots & n \end{pmatrix}.$$

e.g. By prefix reversals

$$(\underline{5\ 4\ 1}\ 6\ 3\ 2)$$
$$(\underline{1\ 4\ 5\ 6}\ 3\ 2)$$
$$(\underline{6\ 5\ 4}\ 1\ 3\ 2)$$
$$(\underline{4\ 5\ 6\ 1}\ 3\ 2)$$
$$(\underline{1\ 6\ 5\ 4\ 3\ 2})$$
$$(\underline{2\ 3\ 4\ 5\ 6}\ 1)$$
$$(\underline{6\ 5\ 4\ 3\ 2\ 1})$$
$$(1\ 2\ 3\ 4\ 5\ 6)$$

The first element must be involved in the reversals.

(Takes more prefix transpositions!)

$\rho(1,j)$

Take $i=1$.

An example

The Pancake problem (The prefix reversal problem)

π: $(\underline{1\ 7\ 5}\ 3\ 6\ 4\ 2\ 8)$

$\Downarrow$

$(\underline{5\ 7\ 1\ 3\ 6}\ 4\ 2\ 8)$

$\Downarrow$

$(\underline{6\ 3\ 1}\ 7\ 5\ 4\ 2\ 8)$

$\Downarrow$

$(\underline{1\ 3\ 6\ 7\ 5\ 4}\ 2\ 8)$

$\Downarrow$

$(\underline{4\ 5\ 7\ 6}\ 3\ 1\ 2\ 8)$

$\Downarrow$

$(\underline{6\ 7}\ 5\ 4\ 3\ 1\ 2\ 8)$

$\Downarrow$

$(\underline{7\ 6\ 5\ 4\ 3\ 1\ 2}\ 8)$

$\Downarrow$

$(\underline{2\ 1}\ 3\ 4\ 5\ 6\ 7\ 8)$

$\Downarrow$

$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)$

$rd(\pi) \le prd(\pi) \le 8$

rd → reversal distance

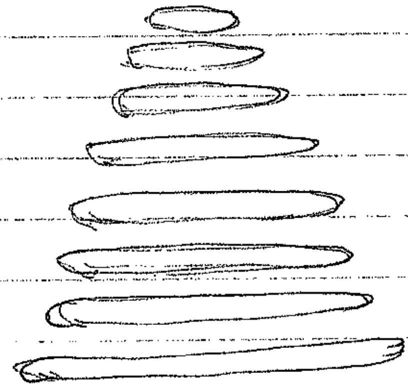
$\ge sb(\pi)/2$

sb → strong breakpoints

(不算 back-adjacency)

1 ~ 8 代表直径为分别为 1~8 英吋的 pancakes.

$\Downarrow$



← 翻面

$$1\ 7\ \underline{5\ 3\ 6}\ 4\ 2\ 8$$
$$\Downarrow$$
$$1\ 7\ 6\ \underline{3\ 5}\ 4\ 2\ 8$$
$$\Downarrow \qquad rd(\pi) \leq 4.$$
$$1\ 7\ 6\ 5\ \underline{3\ 4}\ 2\ 8$$
$$\Downarrow$$
$$1\ \underline{7\ 6\ 5\ 4\ 3\ 2}\ 8 \qquad sb(\pi) = 8$$
$$\Downarrow$$
$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 \qquad \cdot 1 \cdot 7 \cdot 5 \cdot 3 \cdot 6 \cdot 4 \cdot 2 \cdot 8$$

$$rd(\pi) = 4.$$

<u>Remark</u> $rd(\pi)$ is less than $prd(\pi)$ in general.

<u>Reference</u>

William H. Gates and Christos H. Papadimitriou,

Bounds for sorting by prefix reversal, DM 27 (1979), 45-57.

<u>Results</u>

Let π be a permutation in S_n.

$\max\{prd(\pi) \mid \pi \in S_n\}$ is denoted by $prd(n)$.

$$\Downarrow$$

$$\begin{cases} (1) & prd(n) \leq (5n+5)/3 \text{ and} \\ (2) & prd(n) \geq \frac{17n}{16} \text{ for } n \text{ a multiple of } 16. \end{cases}$$

Gollan permutations

(*) The permutations with largest reversal distance: $n-2$.

$$\begin{cases} \pi_1 = (1\ 3\ 5\ \cdots\ n-1\ n\ \cdots\ 6\ 4\ 2), & n \text{ is even;} \\ \pi_2 = (1\ 3\ 5\ \cdots\ \underline{n}\ n-1\ \cdots\ 6\ 4\ 2), & n \text{ is odd.} \end{cases}$$

e.g. $n = 10$

$$\pi_1 = (1\ 3\ 5\ 7\ 9\ 10\ 8\ 6\ 4\ 2),$$

$n=11$ $\quad \pi_2 = (1\ 3\ 5\ 7\ 9\ 11\ 10\ 8\ 6\ 4\ 2).$

$$(1\ 3\ 5\ 7\ \underline{9\ 10}\ 8\ 6\ 4\ 2)$$

1

$$(1\ 3\ 5\ 7\ \underline{10\ 9\ 8}\ 6\ 4\ 2)$$

2

$$(1\ 3\ 5\ \underline{7\ 8\ 9\ 10}\ 6\ 4\ 2)$$

3

$$(1\ 3\ 5\ \underline{10\ 9\ 8\ 7\ 6}\ 4\ 2)$$

4

$$(1\ 3\ \underline{5\ 6\ 7\ 8\ 9\ 10}\ 4\ 2)$$

5

$$(1\ 3\ \underline{10\ 9\ 8\ 7\ 6\ 5\ 4}\ 2)$$

6

$$(1\ \underline{3\ 4\ 5\ 6\ 7\ 8\ 9\ 10}\ 2)$$

7

$$(1\ \underline{10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2})$$

8

$$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10)$$

$(1\ 3\ 5\ 7\ 9\ \underline{11\ 10}\ 8\ 6\ 4\ 2)$

1 $(1\ 3\ 5\ 7\ \underline{9\ 10\ 11}\ 8\ 6\ 4\ 2)$

2 $(1\ 3\ 5\ 7\ \underline{11\ 10\ 9\ 8}\ 6\ 4\ 2)$

3 $(1\ 3\ 5\ \underline{7\ 8\ 9\ 10\ 11}\ 6\ 4\ 2)$

4 $(1\ 3\ 5\ \underline{11\ 10\ 9\ 8\ 7\ 6}\ 4\ 2)$.

5 $(1\ 3\ \underline{5\ 6\ 7\ 8\ 9\ 10\ 11}\ 4\ 2)$

6 $(1\ 3\ \underline{11\ 10\ 9\ 8\ 7\ 6\ 5\ 4}\ 2)$

7 $(1\ \underline{3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11}\ 2)$

8 $(1\ \underline{11\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2})$

9 $(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)$

Can you find another permutation π, s.t. $rd(\pi) \geq n-2$?

Problem

Let $ed(\pi)$ be the edit-distance such that we can use transpositions and reversals for editing permutations π. Clearly, $ed(\pi) \leq \min\{td(\pi), rd(\pi)\}$. Estimate $ed(\pi)$.

(∘) We can also consider the corresponding distance of signed permutations. In fact, in real applications, modeled by signed permutation is much better due to the nature of orientation of genes.

(∗∘) Interesting readers may refer to the book "Combinatorics of Genome Rearrangements".

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Here, we introduce several other variants about distance.

1. Block interchange distance (Blocks of the same length)

$\beta(i, j, k, l)$

$$(\pi_1\ \pi_2 \cdots \boxed{\pi_i \cdots \overset{\pi_{j-1}}{\downarrow}}\ \pi_j \cdots \boxed{\pi_k \cdots \overset{\pi_{l-1}}{\downarrow}}\ \pi_l \cdots \pi_n)$$

$$\Downarrow$$

$$(\pi_1\ \pi_2 \cdots \pi_{i-1} \boxed{\pi_k \cdots \pi_{l-1}}\ \pi_j \cdots \pi_{k-1} \boxed{\pi_i \cdots \pi_{j-1}}\ \pi_l \cdots \pi_n)$$

$$= \begin{pmatrix} 1 & 2 & \cdots & i-1 & \boxed{i \cdots j-1} & j & \cdots & k-1 & k & \cdots & l-1 & l & l+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & \boxed{k \cdots l-1} & j & \cdots & k-1 & i & \cdots & j-1 & l & l+1 & \cdots & n \end{pmatrix}$$

這裏交換的 blocks 長度一樣!

$bid(\pi)$: block interchange distance

<u>Proposition</u> Let $c(G(\pi))$ be the number of cycles in the cycle graph of π. Then, $\forall\, \pi \in S_n$,

$$bid(\pi) = \frac{n+1-c(G(\pi))}{2}$$

<u>Proof.</u> The proof follows by adding two cycles after a block interchange. (Equivalently, add two more adjacency.)

e.g. $(5\ \boxed{4\ 1}\ \boxed{6\ 3}\ 2)$

$\Downarrow$

$(\boxed{5\ 6}\ 3\ 4\ \boxed{1\ 2})$

$\Downarrow$

$(1\ 2\ 3\ 4\ 5\ 6)$

<u>Element interchange distance</u>

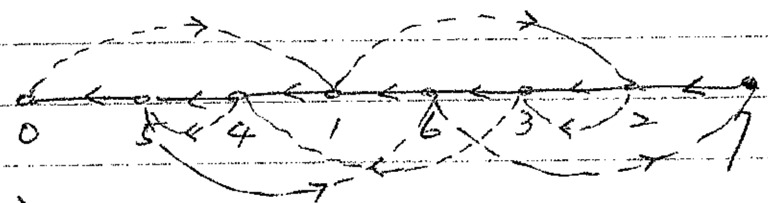
This is clearly a special case of block change, each block has exactly one element.

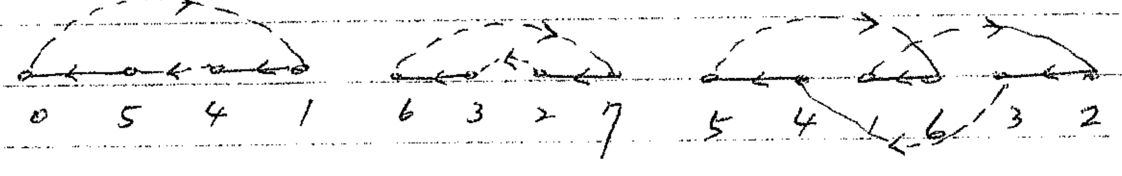
---

5 6 3 4 1 2

<u>Review</u> $\pi = (5\ 4\ 1\ 6\ 3\ 2)$ $n=6$

$$bid(\pi) = \frac{7-3}{2} = 2$$





exc(π) : element exchange distance

(•) This distance is different that of "insertion leading elements".

(••) If we restrict those valid exchanges to adjacent elements, then we have adjacent exchanges. This idea is equivalent to "fixed-length reversals" with length "2".

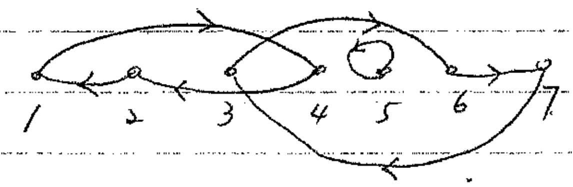
$exc(\pi) \le n-1$ (Trivial fact). $\pi \equiv (\pi_1 \pi_2 \cdots \pi_n)$

In fact, we can do much better. Let δ_π be the number of π_i's such that $\pi_i = i$, $i = 1, 2, \cdots, n$. Then, $exc(\pi) \le n - \delta - 1$.

Proposition Let $\Gamma(\pi)$ be the Γ-graph of π and $c(\Gamma(\pi))$ be the number of cycles in $\Gamma(\pi)$. Then, $exc(\pi) = n - c(\Gamma(\pi))$.

Proof. The idea follows from looking at the following example.

$$\pi = (4\ 1\ 6\ 2\ 5\ 7\ 3)$$

$\Gamma(\pi)$: 

Two steps for 3-cycle and k steps for $(k+1)$-cycle

$(4\ 1\ 6\ 2\ 5\ 7\ 3)$

$\Downarrow$ $1 \to 4^{th}$ position

$(4\ 2\ 6\ 1\ 5\ 7\ 3)$

$(1\ 2\ 6\ 4\ 5\ 7\ 3)$

$(1\ 2\ 7\ 4\ 5\ 6\ 3)$.

$(1\ 2\ 3\ 4\ 5\ 6\ 7)$.

If $\Gamma(\pi)$ contains cycles of length $k_1, k_2, \cdots, k_t$ such that $\sum_{i=1}^{t} k_i = n$, then it takes $\sum_{i=1}^{t}(k_i - 1)$ steps to arrive "i".

This value is equal to $n-t$. Hence, $exc(\pi) \leq n - c(\Gamma(\pi))$.

On the other hand, it takes at least $k_i - 1$ steps to obtain k_i positions such that $\pi_j = j$ for j in that cycle of length k_i. Thus, $exc(\pi) \geq n - c(\Gamma(\pi))$. ∎

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Note

Sorting by Insertion of Leading Elements

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Consider the operation on permutations consisting of removing the leading element and inserting it somewhere in the string. The number of such operations required to sort a permutation σ of $\{1,..., n\}$ into increasing order is $n-k$, where k is the largest integer such that the last k entries of σ form an increasing sequence. There is a deterministic version of the problem, in which the leading element is always inserted into the position equal to its value, and the process ends when 1 reaches the front. The permutation requiring the greatest number of steps to termination is $23 \cdots n1$, and it requires $2^{n-1}-1$ steps. © 1987 Academic Press, Inc.

1. INTRODUCTION

Some people sort canceled checks each month by the following procedure. They hold the stack of checks in one hand, and with the other hand they remove the check at the front and insert it at some appropriate later position in the stack. For example, they may build up a sorted permutation of the checks seen thus far at the back of the stack. In this way, they can always complete the sorting in one pass through the checks. In fact, this procedure uses the minimum number of operations. We also consider a deterministic version of the problem. Both versions arose as variants of the corresponding versions of the pancake problem.

The pancake problem, originally prosed by Goodman under the pseudonym Harry Dweighter [1], asks for the worst-case complexity of sorting permutations by prefix reversal; i.e., the elementary operations are

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flipping a top portion of a stack of pancakes. Gates and Papadimitriou [4] showed that the worst-case complexity of sorting n-element permutations with this operation is between $17n/16$ and $5n/3$. Related questions can be generated by allowing flips of non-leading strings or by requiring the pancakes to finish face up (the so-called "burnt pancake problem"). A deterministic version of the problem arose in England and was circulated by Berman and Klamkin. In this the number of items to be flipped at the next step is precisely the value of the top element. The procedure terminates when element 1 reaches the top. Knuth [2] found an upper bound of $1+F_n$ for the longest flipping sequence on n-element permutations, where F_n is the nth Fibonacci number. The answer is thought by many to be quadratic, but no non-linear lower bound is known.

A systematic way of obtaining variations of both the deterministic and sorting versions is to consider other operations applied to the beginning of the permutations. Canfield and Robbins [3] have considered a number of such operations. In this note we study both versions for the operation of *head insertion*. This is the operation of inserting the leading element at some other position in the stack. In the sorting version, the leading element may be inserted at any position of choice. In the deterministic version, the leading element must move to position i if its value is i. The elements from position 2 through the destination of the leading element slide forward one position. We show that the number of insertions required to sort a permutation can be computed exactly by examining the back part of the permutation, ($n-1$ in the worst case), while the number of insertions in the deterministic version is $2^{n-1}-1$ in the worst case.

2. The Sorting Problem

THEOREM 1. *The number of insertions required to sort a permutation σ by head insertions is $n-k$, where k is the largest integer such that the last k entries of σ form an increasing subsequence.*

Proof. To sort σ with this many insertions, enlarge the increasing sequence at the rear of σ by one with each insertion. That is, insert the leading element in the unique position such that it will enlarge that increasing sequence. After $n-k$ steps, the permutation will end with an increasing sequence of size n, which can only be the fully sorted permutation.

The lower bound is equally easy. To obtain the sorted permutation from σ, the order of σ_{n-k} and σ_{n-k+1} must be changed. Since changes occur only by stripping elements off the top, this means that all the elements preceding σ_{n-k+1} must be stripped, so that at least $n-k$ operations must be performed. ∎

The theorem applies equally well to achieving an arbitrary permutation; simply define "increasing sequence" with respect to the target permutation. We also note that the number of permutations requiring precisely $n-k$ insertions is $k\binom{n}{k+1}(n-k-1)! = kn!/(k+1)!$ for $k<n$ and 1 for $k=n$, which gives a combinatorial interpretation of the identity $\sum_{k=1}^{n} k/(k+1)! = 1 - 1/n!$.

3. The Deterministic Problem

A prefix reversal changes only one adjacency in a permutation, whereas a head insertion changes two. From this point of view one might expect deterministic insertion to terminate more quickly than deterministic prefix reversal. However, this is not the case.

Lemma. *Under deterministic head insertion, the value $n-k$ appears at the front at most 2^k times.*

Proof. By induction on k. For $k=0$, note that the value n goes immediately to the rear if it reaches the front, and nothing can ever make it move again. Now consider $n-k$ for $k>0$. After each time it appears at the front, a higher value must appear at the front to enable it to start moving forward again. Counting the first appearance, this bounds its appearances by $1+\sum_{l=0}^{k-1} 2^l = 2^k$. ∎

This lemma immediately yields the correct upper bound, since there must be some value (exceeding 1) at the front before each step. Summing over the appearances of each possible value, we get a bound of $\sum_{k=0}^{n-2} 2^k = 2^{n-1}-1$. However, we can give more detail, which in addition yields the unique permutation achieving the bound.

Theorem 2. *If σ is a permutation with $\sigma_k = 1$, then deterministic insertion on σ takes at most $2^{k-1}-1$ steps. This is best possible, and is attained only by the permutations of the form $23\cdots(k-1)\,x1x\cdots x$, where x denotes any value at least k. Furthermore, if the process is executed on such a permutation, then the numbers $123\cdots(k-1)$ appear at the front in order when the process terminates.*

Proof. By induction on k. For $k=1$, 1 is already at the front, and we get 0 steps. For $k>1$, consider what is required to start item 1 moving forward. An item with value at least k must reach the front. To get there before 1, it must precede 1 in σ. Let y be the earliest such item in σ, say at position $j<k$. Since y reaches the front before element 1 moves, the sequence of head values before 1 moves will be the same in the permutation

σ' obtained from σ by interchanging elements y and 1. By induction, there are at most $2^{j-1}-1$ steps (on σ) until y reaches the front. The next step moves 1 from position k to position $k-1$. By induction, at most $2^{k-2}-1$ steps now remain. Hence the total number of steps is at most $2^{j-1}-1+1+2^{k-2} \leqslant 2^{k-1}-1$.

To achieve the bound, a permutation must take the maximum number of steps in each phase. This requires first that the only item with value at least k appear in position $j=k-1$. Next, by induction moving this element to the front will take $2^{k-2}-1$ steps only if σ begins as $23\cdots(k-1)\,x1$. If this is so, then applying induction to the permutation σ guarantees that after $2^{k-2}-1$ steps σ turns into the permutation beginning as $x23\cdots(k-1)\,1$. After the next step, its first $k-1$ positions are $23\cdots(k-1)\,1$. Applying the induction hypothesis again says that $2^{k-2}-1$ more steps produce a permutation beginning as $12\cdots(k-1)$. This statement completes the induction step. ∎

In fact, if j is the position of the earliest number at least k, then it takes at most 2^{j-1} steps to move 1 from position k to position $k-1$. Hence a more detailed analysis can be given, but it is not clear that determination of the number of steps required is in general easier than running the procedure. Indeed, it is possible that determining the number of steps this procedure takes on an arbitrary permutation is NP-hard.

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