Rearrangement problems on Permutations

Review

Let α and β be two permutations of S_n. Then, the composition of α and β, denoted by $\alpha \circ \beta$, maps i into (onto) $(\alpha\circ\beta)(i) = \alpha(\beta(i))$. For example, $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, (or $\alpha = (3142)$, $\beta = (1342)$ in short, then

$$\alpha\circ\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} = (3421).$$

(o) If $\alpha \circ \beta = \gamma$, then $\beta = \alpha^{-1}\circ\gamma$. (Notice that S_n is a symmetric group with "operation", composition.) (o) $(\alpha\circ\beta)^{-1} = \beta^{-1}\circ\alpha^{-1}$.

Problem

Given any two permutations π and σ in G (say S_n) and a set S of generators of G, find a minimum number permutations in S, $\rho_1, \rho_2, \cdots, \rho_k$ such that $\pi\circ\rho_1\circ\rho_2\circ\cdots\circ\rho_k = \sigma$, equivalently,

$$\sigma^{-1}\circ\pi = \underbrace{\rho_k^{-1}\circ\rho_{k-1}^{-1}\circ\cdots\circ\rho_1^{-1}}_{\text{Minimum length factorization.}}$$

(*) Finding a minimum length factorization of permutations is an NP-hard problem. [Even and Goldreich, J. Algorithm 2 (1981), 311–313.]

<u>Definition</u> (Invariant distance)

Given a set S of generators of a permutation group, A distance between two permutations π and σ (w.r.t. S) can be defined as the minimum number of generators $\rho_1, \rho_2, \cdots, \rho_k$ (in S) such that $\pi \circ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_k = \sigma$, denoted by $d_S(\pi, \sigma) = k$.

(*) By the definition of "metric", d_S is indeed a metric.

<u>Definition</u> (Left-invariant)

A distance d on G is left-invariant if for all π, σ, τ in G, $d(\pi, \sigma) = d(\tau \circ \pi, \tau \circ \sigma)$.

(**) $\pi \circ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_k = \sigma \Rightarrow (\sigma^{-1} \circ \pi) \circ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_k = \iota$ (identity).

$d(\pi, \sigma) = d(\sigma^{-1} \circ \pi, \iota)$

$d(\pi, \iota) = d(\pi^{-1}, \iota) = d(\iota, \pi)$

(*) d_S is left-invariant.

Let $d_S(\pi, \sigma) = k$ and $\pi \circ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_k = \sigma$. Then, $(\tau \circ \pi) \circ \rho_1 \circ \cdots \circ \rho_k = (\tau \circ \sigma)$.
$\Rightarrow d_S(\tau \circ \pi, \tau \circ \sigma) \le k = d_S(\pi, \sigma)$.
Apply τ^{-1} to obtain $d_S(\pi, \sigma) \le d_S(\tau \circ \pi, \tau \circ \sigma)$.

<u>Definition</u> (Sorting Sequence)

Given a permutation π, a sequence of rearrangements $\rho_1, \rho_2, \cdots, \rho_k$ is called a <u>sorting sequence</u> for π if $\pi \circ \rho_1 \circ \rho_2 \circ \cdots \circ \rho_k = \iota$.

How to find the distance?

Definition (Cayley graph)

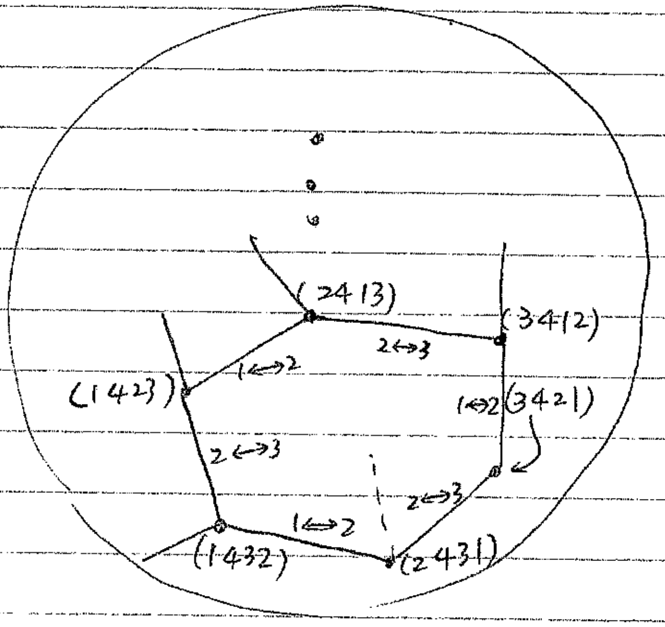
Given a set S of generators of a permutation group $\mathcal{G}$, the Cayley graph associated with $\underline{(S, \mathcal{G})}$ is the graph whose vertices are the elements of $\mathcal{G}$ and two vertices are adjacent if the corresponding permutations can be transformed into one another using an element of S.

Example

$\mathcal{G} = S_4$, $S = \{$exchange of adjacent elements $1 \leftrightarrow 2, 2 \leftrightarrow 3, 3 \leftrightarrow 4\}$

(Bubble sort)

$G \underset{def}{=} \mathrm{Cayley}(S_4, S)$:



3-regular graph of order 24.

(◦) The distance of two permutations is equal to the shortest distance in $G \underset{def}{=} \mathrm{Cayley}(S_4, S)$.

Date / /

Reversals

$$\pi = (\pi_1\ \pi_2 \cdots \pi_{i-1}\ \boxed{\pi_i\ \pi_{i+1} \cdots \pi_j}\ \pi_{j+1} \cdots \pi_n)$$

permutation

$$\Downarrow$$

$$(\pi_1\ \pi_2 \cdots \pi_{i-1}\ \boxed{\pi_j\ \pi_{j-1} \cdots \pi_i}\ \pi_{j+1} \cdots \pi_n)$$

Let $\rho = (1\ 2 \cdots i-1\ j\ j-1 \cdots i+1\ i\ j+1\ j+2 \cdots n)$.

Then $\pi \circ \rho =$

e.g. $(1\ 3\ 5\ 2\ 4) \circ (1\ 4\ 3\ 2\ 5)$ ρ

$= (1\ 2\ 5\ 3\ 4)$

Signed permutation

$$\rho = (1\ 2 \cdots i-1\ \boxed{-j\ -j+1 \cdots -i-1\ -i}\ j+1 \cdots n)$$

$$\pi \circ \rho = (\pi_1\ \pi_2 \cdots \pi_{i-1}\ \boxed{-\pi_j\ -\pi_{j-1} \cdots -\pi_{i+1}\ -\pi_i}\ \pi_{j+1} \cdots \pi_n)$$

Problem Use only reversals to find the minimum length factorization.

$$\pi \rightarrow \rightarrow \rightarrow \rightarrow \cdots \rightarrow i$$

$(1\ 2\ 3 \cdots n)$

(at least)

Observation $\pi_1 \neq 1$ one reversal is needed!

$\pi_n \neq n$ 〃

$\pi_{j+1} > \pi_j$ 〃

⋮

(◦) Similarity between permutations

(◦) If a group of genes appears consecutively in several species, then they must have been present in the same order in the ancestral species, and were not separated during evolution.

Definition (Linear extension)

The linear extension of (signed or unsigned) permutation π of $\{1, 2, \cdots, n\}$ is the permutation of $\{0, 1, 2, \cdots, n, n+1\}$ defined by $\pi^l = (0\ \pi_1\ \pi_2 \cdots \pi_n\ n+1)$. (If signed permutations are considered, then we let "0" be of positive sign.)

Example (0 4 8 9 7 6 5 1 3 2 10) is a linear extension of (4 8 9 7 6 5 1 3 2).

Definition (Break "points")

Let π^l be the linear extension of π. A point of π is an ordered pair (π_i^l, π_{i+1}^l) (denoted by $\pi_i^l \circ \pi_{i+1}^l$ in short), $0 \le i \le n$.

- If $\pi^{l}_{i+1} = \pi^{l}_{i} + 1$, then it is called an adjacency (point);
- If $\pi^{l}_{i+1} = \pi^{l}_{i} - 1$, then it is called a reverse adjacency (point);
- If it is not an adjacency, then it is called a breakpoint;
- If it is neither an adjacency nor a reverse adjacency, then it is called a strong breakpoint.

e.g. $0 \cdot 4 \cdot 8 \cdot 9 \cdot 7 \cdot 6 \cdot 5 \cdot 1 \cdot 3 \cdot 2 \cdot 10$

adjacency

reverse adjacency
(break points)

All the others are strong break points.

$p(\pi)$: The # of points

$bp(\pi)$: The # of breakpoints

$sb(\pi)$: The # of strong breakpoints

(*) Breakpoints yield a first example of an evolutionary distance between genomes: the "breakpoint distance" between two permutations. (The difference of adjacencies)

π, σ

$$\boxed{bp(\pi,\sigma) =_{def} bp(\sigma^{-1}\circ\pi)}$$

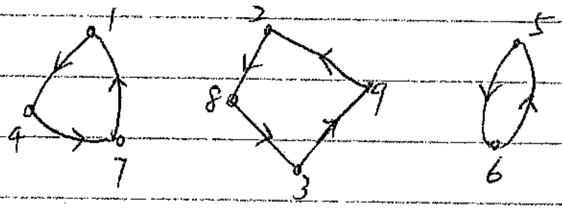
(breakpoint distance

Cycle graph of a permutation

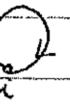
It is known in Algebra that each permutation can be written as a composition of disjoint cycles.

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 8 & 9 & 7 & 6 & 5 & 1 & 3 & 2 \end{pmatrix}$

$= (1\,4\,7)(2\,8\,3\,9)(5\,6)$ ⟵ Cycle decomposition graph



The graph (digraph) which represents the cycles of a permutation is called a cycle decomposition graph of the permutation.

(i) We consider (i) a cycle (loop), .

(ii) If the cycle decomposition graph of a permutation is connected (strongly), then the permutation is called a circular permutation, denoted by $[x_1 x_2 \cdots x_n]$. So, a circular permutation has n equivalent forms $(x_1 x_2 \cdots x_n), (x_2 x_3 \cdots x_n x_1), \cdots, (x_n x_1 \cdots x_{n-1})$.

Cycle graph of π

Definition (Cycle graph)

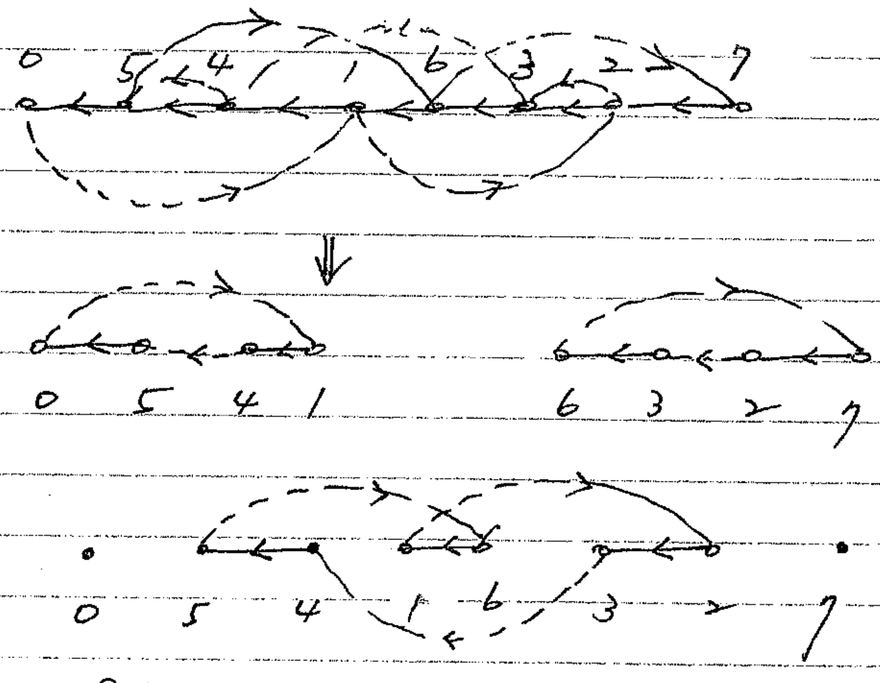
The cycle graph of a permutation π of $\{1, 2, \cdots, n\}$ is the directed graph $G(\pi)$ with vertex set $\{0, 1, 2, \cdots, n, n+1\}$ and whose arcs are :

(1) black arcs (reality) (π^l_i, π^l_{i-1}) for $1 \le i \le n+1$, and

(2) gray (desired) arcs $(i, i+1)$ for $0 \le i \le n$.

Example $\pi = (5\ 4\ 1\ 6\ 3\ 2)$, π^l : linear extension of π

$G(\pi)$:



alternating cycles ← The # of, $c(G(\pi))$.

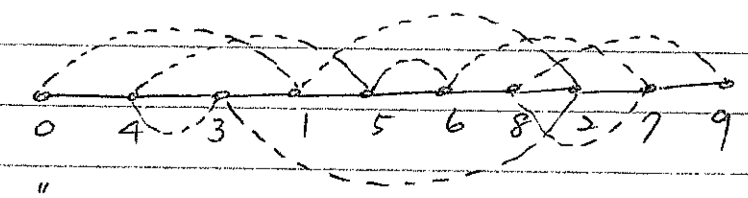
Definition (Breakpoint graph)

The breakpoint graph of a permutation π is the undirected graph $BG(\pi)$, whose vertex set is the vertex set of the cycle graph of π, and whose edges are the arcs of the cycle graph of π taken without their orientation.

e.g. circular permutation $(4\,3\,1\,5\,6\,8\,2\,7)$

extension $0 \circ 4 \circ 3 \circ 1 \circ 5 \circ 6 \circ 8 \circ 2 \circ 7 \circ 9$

$\tilde{G}$:



Incident edges " $\cdots$ "

$\tilde{G}$ is obtained by adding incident edges $01, 12, 23, \cdots, 89$.

$\tilde{G}$ is an eulerian graphs.

$\tilde{G}$ can be decomposed into alternating (circuits) cycles :

$(0,4,5,1)$, $(5,6)$, $(4,3)$, $(6,8,9,7)$, $(3,1,2,8,7,2)$

Let $c^*(BG(\pi))$ denote the number of cycles in a maximal alternating cycle decomposition of $BG(\pi)$.

The goal of finding minimum length factorization can be accomplished by using both the cycle graph and breakpoint graph of π. Though not able to find an explicit solution, we are able to estimate the upper bounds, hopefully a tight bound.

Lecture 7 (Continued)

Transposition Distance

(•) The term "transposition" comes from biology and refers to "transposons", which are sequences of DNA that can be displaced in a genome.

Note : Algebraic transposition is a 2-cycle in a permutation, for $(i\ j)$ which represents $i \to j$ and $j \to i$.

Definition (Interval of permutation)

An interval (or segment) of a permutation $\pi = (\pi_1 \pi_2 \cdots \pi_n)$ is a set $\{|\pi_a|, |\pi_{a+1}|, \cdots, |\pi_b|\}$ with $1 \le a \le b \le n$. The absolute value is meaningful for signed permutation, and the elements π_a and π_b are the extremities of the interval.

⇓

Definition (Transposition $\tau(i,j,k)$)

For any permutation π in S_n, the transposition $\tau(i,j,k)$ with $1 \le i < j < k \le n+1$ applied to π exchanges the intervals determined respectively by i and $j-1$ and by j and $k-1$, transforming π into $\pi \circ \tau(i,j,k)$. Therefore,

$$\tau(i,j,k) = \begin{array}{c} \qquad\qquad\quad \pi_{i-1}\ \pi_i \qquad\qquad\quad \pi_{j-1}\ \pi_j \qquad\qquad \pi_{k-1}\ \pi_k \\ \begin{pmatrix} 1 & 2 & \cdots & i-1 & i & \cdots & j-2 & j-1 & j & j+1 & \cdots & k-1 & k & \cdots & n \\ 1 & 2 & \cdots & i-1 & j & j+1 & \cdots & k-1 & i & \cdots & j-2 & j-1 & k & \cdots & n \end{pmatrix} \end{array}.$$

(*) 把由 i 到 $j-1$ 的 interval 移到 $k-1$ 的後面, k 的前面。
(插入於 $k-1$ 與 k 之間).

(∘) We use $td(\pi)$ to denote the transposition distance of π, i.e., after $td(\pi)$ transpositions, we end it up with ι (identity).

e.g. $\pi = (5\ 1\ 3\ 2\ 4)$

$\pi \circ \tau(3,4,5) = (5\ 1\ \boxed{2}\ \boxed{3}\ 4)$

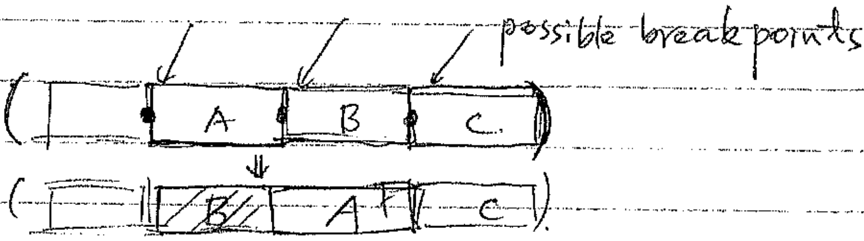
e.g. $\pi = (\boxed{4\ 1}\ \boxed{6\ 2}\ \boxed{5}\ 7\ 3)$, $\tau(3,5,6)$

$$\pi \circ \tau(3,5,6) = (4\ 1\ 5\ \overset{3\ \ 4}{\boxed{6\ 2}}\ 7\ 3)$$

$\uparrow$ $j-1 = k-1$

Proposition Let the number of breakpoints of π be $bp(\pi)$. Then, $td(\pi) \geq \frac{bp(\pi)}{3}$.

Proof.



一次最多少三个 b.p.!

Breakpoints $\begin{cases} \text{ascents} & \pi_{i+1} > \pi_i \quad (\text{But, not } \pi_{i+1} = \pi_i + 1) \\ \text{decents} & \pi_{i+1} < \pi_i \end{cases}$

(∘) $des(\pi)$: The # of decents in π.

Proposition $\forall\ \pi \in S_n$, $td(\pi) \geq \max\left\{ \frac{des(\pi)}{2}, \frac{des(\pi^{-1})}{2} \right\}$.

Proof. It suffices to claim that the # of decents decrease only at most 2. Suppose not.

Let $\pi_{i-1} = a$, $\pi_i = b$, $\pi_{j-1} = c$, $\pi_j = d$, $\pi_{k-1} = e$ and $\pi_k = f$. Moreover, $a > b$, $c > d$ and $e > f$. Following the transposition $\tau(i, j, k)$, we have $a - d$; $e - b$; $c - f$; Also, $a < d$, $e < b$ and $c < f$. But, this is not possible. If $a < d$ and $e < b$, then $c > d > a > b > e > f$. $\rightarrow\leftarrow$ If $a < d$ and $c < f$, then $b < a < d < c < f < e$. $\rightarrow\leftarrow$ Finally, if $e < b$ and $c < f$, then $a > b > e > f > c > d$. $\rightarrow\leftarrow$

Since the distance between π and i is the same as the distance between i and π^{-1}, and $des(\pi)$ may be different from $des(\pi^{-1})$, the assertion holds. ∎

(◦) A strip of π is a maximal interval of π containing no breakpoints.

(◦) Reduced permutation

$\underline{4\,5\,6} \cdot 3 \cdot 1 \cdot \underline{7\,8} \cdot 2 \longrightarrow (4\ 3\ 1\ 5\ 2)$

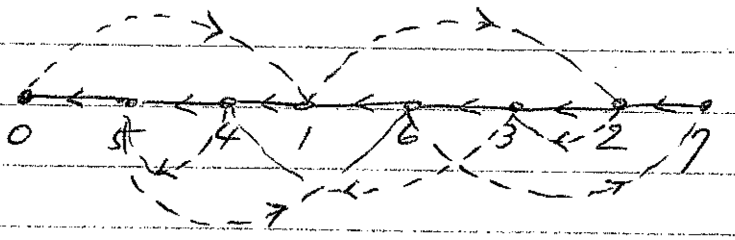
They have the same transposition distance.

Review "cycle graph of π": $G(\pi)$

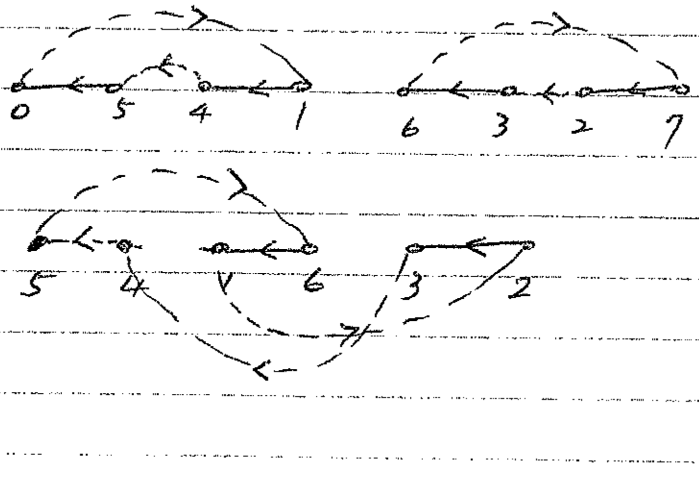
Let $c(G(\pi))$ denote the number of alternating cycles.

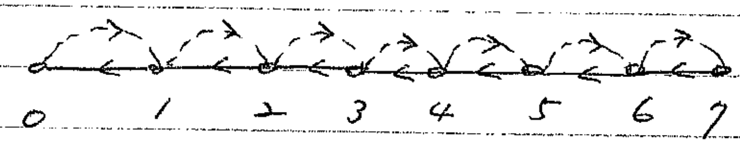
$\pi = (5\ 4\ 1\ 6\ 3\ 2)$

$G(\pi)$:



$c(G(\pi)) = 3$



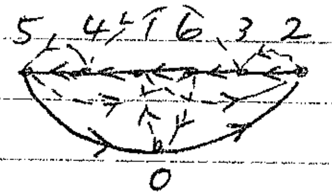


: Identity
most alternating cycles

Cycle graph of linear extension

(*) We can define the cycle graph of π by using circular extension instead of linear extension.

- Example,



"只加0，不加7！"

虛線部份仍照依順序，6接回0。

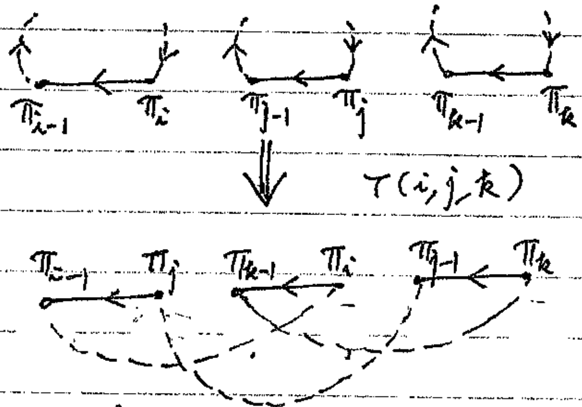
<u>Proposition</u> $\forall\ \pi \in S_n,\ td(\pi) \geq \dfrac{p(\pi) - c(G(\pi))}{2}$.

<u>Proof</u>. It suffices to show that a transposition may increase the number of alternating cycles by at most two. Let $\tau(i,j,k) \underset{def}{=} \tau$ be the transposition, and $\Delta c(\tau) = c(G(\pi\circ\tau)) - c(G(\pi))$ be the number of alternating cycles which are changed. Now, we claim $\Delta c(\tau) = \{-2, 0, 2\}$.

By definition, the vertices of $G(\pi)$ which are involved in the transpositions are $\pi_{i-1}, \pi_i, \pi_{j-1}, \pi_j, \pi_{k-1}$ and π_k. Following the transposition $\tau(i,j,k)$, the black edges (directed) (π_i, π_{i-1}), (π_j, π_{j-1}) and

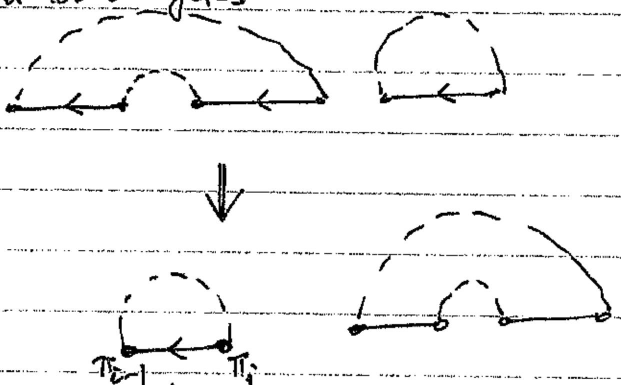
(π_k, π_{k-1}) are removed and adding new black edges (π_j, π_{i-1}), (π_i, π_{k-1}) (in new permutation) and (π_k, π_{j-1}).

Case 1. Three removed edges are in three alternating cycles (of the decomposition of $G(\pi)$).



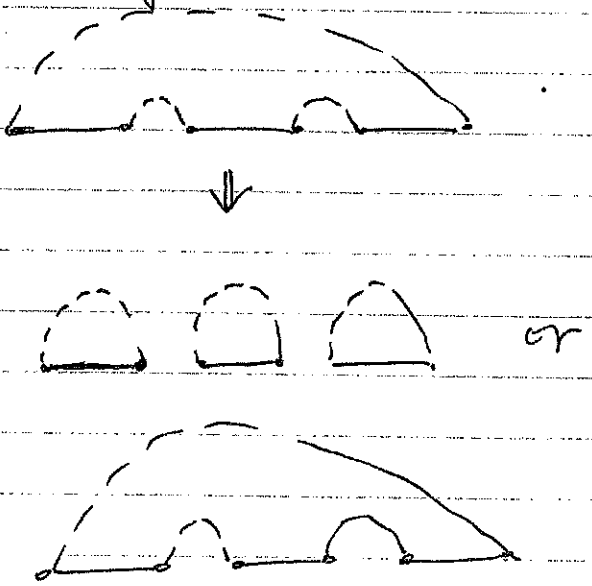
3 cycles
⇓
one cycle

Case 2 In two cycles



Two cycles
⇓
Two cycles

Case 3 In one cycle



one cycle
⇓
Three cycles.
or
One cycle

<u>Definition</u> (Odd or even alternating cycles)

An alternating cycle is said to be odd if it contains an odd number of "black" edges (arcs), and even otherwise.

<u>Example</u> In the decomposition of [0 5 4 1 6 3 2], there are two even cycles and one odd cycle.

(•) The change of odd cycles in a transposition is either −2, 0, or 2.

<u>Proposition</u> $td(\pi) \geq \frac{p(\pi) - c_o(G(\pi))}{2}$ where $c_o(G(\pi))$ denotes the number of odd cycles.

<u>Reference</u>

V. Bafna and P. A. Pevzner, Sorting by transpositions, SIAM J. Discrete Math., Vol. 11, No 2, 224–240.

Upper bounds

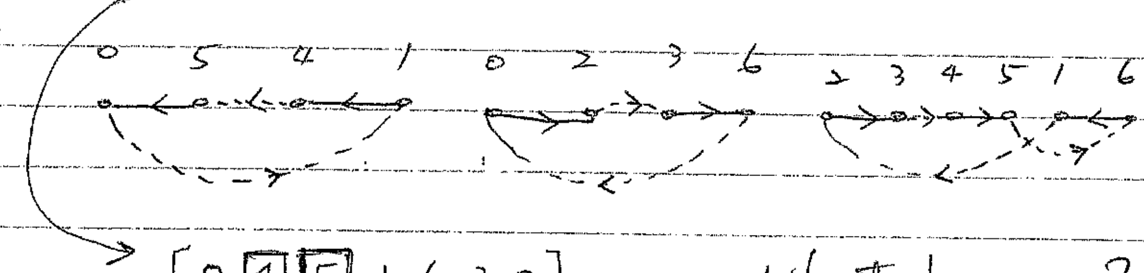
In order to obtain a good upper bound, we need to provide a suitable algorithm to sort the permutations by way of transpositions.

We shall use circular permutations, thus $p(\pi) = n+1$ if $\pi \in S_n$.

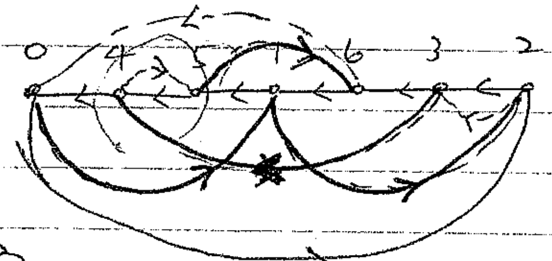
Proposition $td(\pi) \le p(\pi) - c(G(\pi))$.

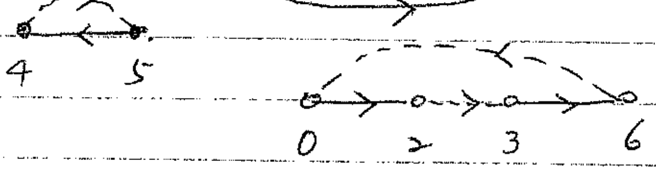
Proof. At each step, we creat one "more" adjacency. Corresponding to this fact, we obtain one more alternating cycle in cycle decomposition of $G(\pi)$. (How?)

Example [0 5 4 1 6 3 2]



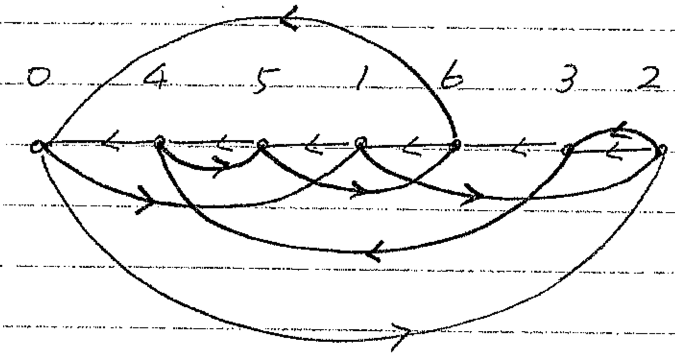
[0 $\boxed{4}$ $\boxed{5}$ 1 6 3 2] What happen?

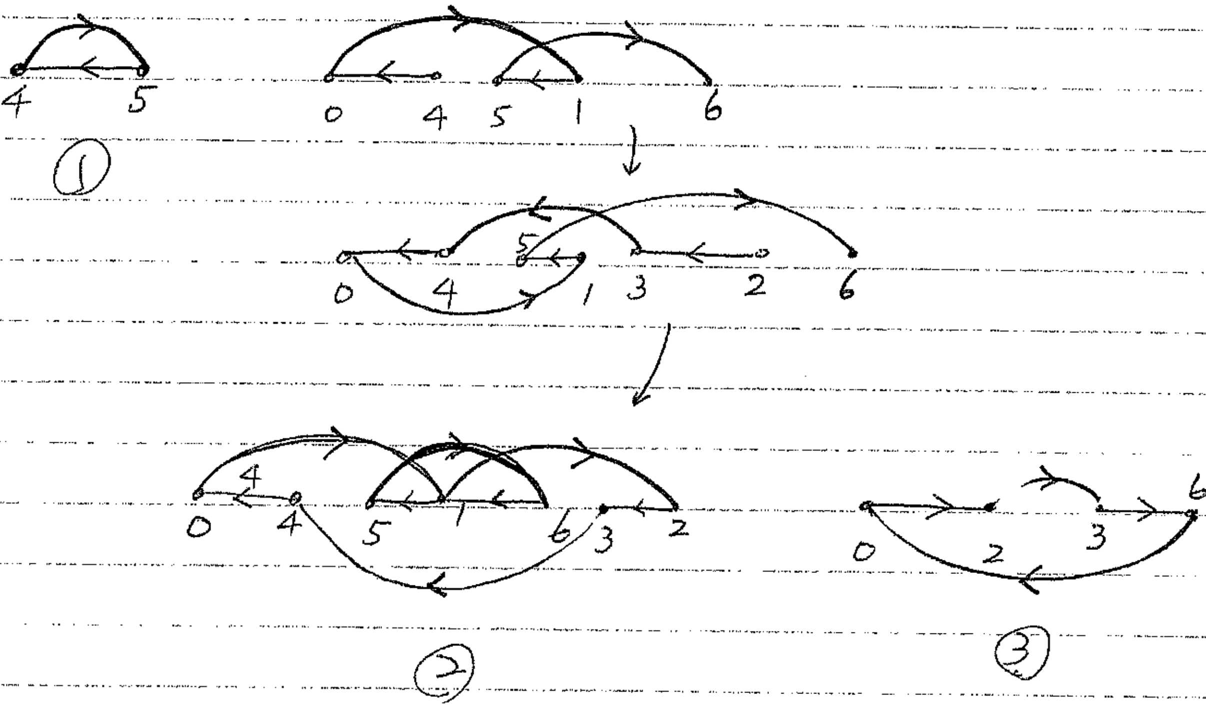




?

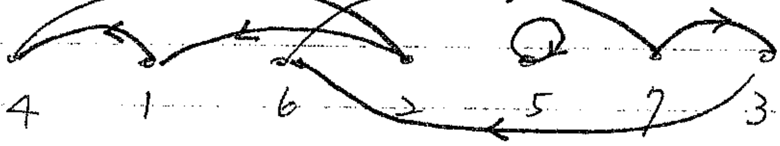
4 0 1 5 6 1 2 3





<u>Definition</u> (Γ-graph)

The Γ-graph of a permutation $\pi \in S_n$ is the digraph $\Gamma(\pi)$ with ordered vertex-set $(\pi_1, \pi_2, \cdots, \pi_n)$ and $(\pi_i, \pi_j) \in A(\Gamma(\pi))$ if $\pi_i = j$.



3 odd directed cycles!

$$\begin{pmatrix} 4 & 1 & 6 & 2 & 5 & 7 & 3 \end{pmatrix}$$
$\pi_1 \ \pi_2 \ \pi_3 \ \pi_4 \ \pi_5 \ \pi_6 \ \pi_7$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 6 & 2 & 5 & 7 & 3 \end{pmatrix}$$

The cycles of $\Gamma(\pi)$ can be obtained by finding the cycle decomposition of π. Here, $\pi = \underline{(142)}\,\underline{(367)}\,\underline{(5)}$.

odd cycles

So, the number of cycles (directed) in $\Gamma(\pi)$ is exactly the number of cycles in "cycle decomposition".
(representation)

<u>Proposition</u>

\# of odd cycles.

For all $\pi \in S_n$, $td(\pi) \leq n - c_o(\Gamma(\pi))$.

<u>Proof.</u> Try!!

(•) We can find the longest increasing subsequence of π to obtain an upper bound.

<u>Proposition</u>

Let $|LIS(\pi)|$ denote the length of a longest increasing sequence. Then, $td(\pi) \leq n - |LIS(\pi)|$.

$n - |LIS(\pi)|$ is also denoted by $ulam(\pi)$, called Ulam's distance.

<u>Proof.</u> We may use transposition to increase $LIS(\pi)$ one by one. ■

(at least one!)

<u>Example</u>

$$4\ 1\ 6\ 2\ 5\ 7\ 3$$

$$\Downarrow$$

$$\boxed{6}$$

$$4\ 1\ 2\ 5\ 6\ 7\ 3$$

$$\boxed{4}$$

$$1\ 2\ 4\ 5\ 6\ 7\ 3.$$

$$\boxed{3}$$

$$1\ 2\ 3\ 4\ 5\ 6\ 7$$

$$4\ 5\ 6\ 7\ \boxed{1\ 2\ 3}$$

From 4 to 7.

<u>Another idea</u>

<u>Definition</u> (Coding a permutation)

Given a permutation $\pi = (\pi_1, \pi_2 \cdots \pi_n)$, the left code of the element π_i of π, denoted by $lc(\pi_i) =_{def}$ ↓ next page

$|\{\pi_j \mid \pi_j > \pi_i$ and $0 \le j \le i-1\}|$, for $1 \le i \le n$. ($\pi_0 = 0$, as an inventron)

Similarly, the right code of π_i, denoted by $rc(\pi_i) =_{def}$ $|\{\pi_j \mid \pi_j < \pi_i$ and $i+1 \le j \le n+1\}|$, for $1 \le i \le n$. ($\pi_{n+1} = n+1$)

(◦) The left code of π is $(lc(\pi_1), lc(\pi_2), \cdots, lc(\pi_n))$ and the right code of π is $(rc(\pi_1), rc(\pi_2), \cdots, rc(\pi_n))$.

<u>Example</u> $\pi = (6\ 3\ 2\ 1\ 4\ 5)$

$$\Downarrow$$

$lc(\pi) = (0, 1, 2, 3, 1, 1)$ and

(∵左边项比該项大的個數!)

$rc(\pi) = (5, 2, 1, 0, 0, 0)$.

(右边项比該项小的個數!)

<u>Example</u> $lc(\mathbf{i}) = rc(\pi) = (0, 0, \cdots, 0)$.

<u>Definition</u> (plateau (複數 plateaux))

Let $\vec{s}$ be a sequence of non-negative integers. A plateau of $\vec{s}$ is any maximal length subsequence of contiguous elements in $\vec{s}$ that have the same nonzero value. The number of plateaux in $\vec{s}$ is denoted by $\tilde{p}(\vec{s})$.

<u>Example</u> $\vec{s} = lc(\pi)$ where $\pi = (6\ 3\ 2\ 1\ 4\ 5)$.

$= (0, \underline{1}, \underline{2}, \underline{3}, \underline{1, 1})$. $\tilde{p}(\vec{s}) = 4$.

$$\vec{s} = rc(\pi) = (\underline{5}, \underline{2}, \underline{1}, 0, 0, 0)$$
$$\Downarrow$$
$$\tilde{p}(\vec{s}) = 3$$

Lemma Given a permutation π, the leftmost (resp. righmost) plateau of $lc(\pi)$ (resp. $rc(\pi)$) can be removed by a transposition to the left (resp. the right) without creating any new plaleaux in the code.

Proof. We prove the left code case. Let the leftmost plateau be from position i to $j-1$. Then, all entries of $lc(\pi_i)$ before position i is "0". That is $\pi_1, \pi_2, \cdots, \pi_{i-1}$ are in increasing order. So, if $lc(\pi_i) = a$, then $\rho(i, j, i-a)$ removes the first plateau from the left. ∎

Example

$$a=1$$
$$\pi = (6\ 3\ 2\ 1\ 4\ 5)$$
$$\Downarrow$$

$i=2, a=1$ $\quad \pi' = (3\ 6\ 2\ 1\ 4\ 5)$

$$lc(\pi') = (0,0,2,3,1,1)$$

$$\pi'' = (2\ 3\ 6\ 1\ 4\ 5)$$
$$lc(\pi'') = (0,0,0,3,1,1)$$

$\vdots$

Proposition $td(\pi) \leq \underbrace{\min\{\tilde{p}(lc(\pi)), \tilde{p}(rc(\pi))\}}_{\tilde{p}(\pi)}$.

Proof. We may choose the one with less plateaux among $lc(\pi)$ and $rc(\pi)$. ∎