Lecture 14, Jan. 5, (2023)

Some ideas in Algebraic Graph Theory. (Many approaches!)

(∘) The adjacency matrix $A(D)$ of a directed graph D is $A(D) = [x_{i,j}]_{n\times n}$ where $V(D) = \{v_1, v_2, \cdots, v_n\}$, $x_{i,j} = 1$ if and only if (v_i, v_j) is an arc of D.

(∘) If D is a graph (instead of digraph), then we view each edge of D as a pair of arcs in opposite directions, and thus $A(D)$ is a symmetric $(0,1)$-matrix.

(∘) If G is a simple graph, then we can define the Laplacian of G, denoted by $L(G) = [l_{i,j}]_{n\times n}$ where $l_{i,i} = \deg_G(v_i)$ and $l_{i,j} = -1 = l_{j,i}$ if $\{v_i, v_j\} \in E(G)$.

(∘) We shall consider $A(G)$ where G is a simple graph in what follows.

(∘) The characteristic polynomial of $A(G)$ is defined as

$$\phi(A,x) =_{def} \phi(G,x) = \det(xI_n - A). \quad (A \doteqdot A(G))$$

(∘) The spectrum of A (matrix) is a list of its eigenvalues, the zeros of $\phi(A,x)$, together with their multiplicities.

(*) We may use automorphism group of a graph, $Aut(G)$, to characterize G. (If possible!)

(**) Almost all graphs G, $Aut(G) = \langle \{i\}, \circ \rangle$, 单位元素群。

Example $G \cong C_4$.

$\phi(C_4, x) = x^4 - 4x^2$; zeros are $2, 0, 0, -2$ (0 is of multiplicity 2).

(•) The largest eigenvalue of a graph G is called the index of G and the spectral radius of G is the maximum value of $\{|\lambda_i| \mid i = 1, 2, \cdots, n$, and λ_i is an eigenvalue of $A\}$. In C_4 case, the index and spectral radius of G are equal.

(•) $Spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{pmatrix}$ and $\sum_{i=1}^{t} m_i = n$.

Examples

1. $Spec(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$. $\phi(K_n, x) = (x-n+1)(x+1)^{n-1}$. $diam(K_n) = 1$

2. $Spec(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$. $diam(K_{m,2}) = 2$

Theorem 84

The diameter of a connected graph G is less than the number of distinct eigenvalues.

Proof. Let r be the number of distinct eigenvalues, let them be $\lambda_1, \lambda_2, \cdots, \lambda_r$. Then $\prod_{i=1}^{r}(x-\lambda_i)$ is the minimal polynomial of A. This implies that a linear combination of ($A^0, A, A^2, \cdots, A^r$ are linearly dependent) $I_n = A^0, A^1, \cdots, A^r$ is the zero matrix. Now, consider diam(G). Let $d(v_i, v_j) = \mathrm{diam}(G) = k$. Then, $A^h(i,j) = 0$ for each $0 \le h < \mathrm{diam}(G)$.

Hence, $A^0, A^1, \cdots, A^k$ are linearly independent. (?) This implies ($A^k(i,j) \neq 0$, but $A^h(i,j) = 0$ for $h < k$.) that $k < r$, i.e, diam(G) is less than the number of distinct eigenvalues. ■

(*) Any subset of an independent set is independent.

<u>Theorem 85</u> For every graph G, $\chi(G) \le 1 + \lambda_{max}(G)$.

<u>Proof.</u> Let $\chi(G) = k$ and H be a vertex critical subgraph of G. That is, $\chi(H) = k$ and for each vertex $v \in V(H)$, $\chi(H - v) = k - 1$. By Theorem 56, $\delta(H) \ge k - 1$.

Now, consider λ_{max}. If λ is an eigenvalue of A(G), then there exists an eigenvector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Let $x_j = \max_{i=1}^{n} \{ x_i \mid \vec{x} = (x_1, x_2, \cdots, x_n) \}$. Then

$$\lambda x_j = (A\vec{x})_j = \sum_{v_i \in N(v_j)} x_i \le \deg_G(v_j) \cdot x_j \le \Delta(G) \cdot x_j .$$

Hence, $\lambda_{max} \le \Delta(G)$ and Theorem 85 is an improvement of Brooks Thm.

On the other hand, $\lambda_{max}(G) \ge \delta(G)$. (?) This implies that

$$k \le 1 + \delta(H) \le 1 + \underline{\lambda_{max}(H)} \le 1 + \underline{\lambda_{max}(G)}. \quad ■$$

$$A(G) : \begin{bmatrix} \boxed{A(H)} & \\ & \end{bmatrix}$$

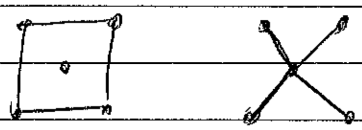
Research problem

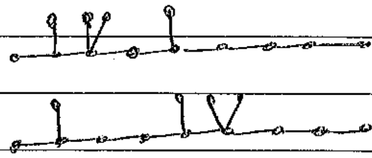
Which graphs are determined by their spectrum?

(◦) Two graphs with the same spectrum are called co-spectral graphs.

大一点的例子

e.g.





$\Downarrow$ spectrum

$$\begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & 1 \end{pmatrix}$$

(◦) Graphs are DS if they are determined by their spectrum.

Conjecture

Almost all graphs are DS? (For trees, almost all trees are non-DS.)

Generalized adjacency matries :

(1) Adjacency matrix A.

(2) $\bar{A} = J - A - I$.

(3) Laplacian matrix $L = D - A$ where $D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$ degrees

(4) Signless Laplacian matrix $\mathcal{L} = D + A$.

(5) Seidel matrix $S = \bar{A} - A = J - 2A - I$.

This course will stop here. Hopefully, the learning experience can provide you the basic knowledge of Graph Theory. For sure, you have learned the skills of "Research" through working on exercises. Keep Moving Forward!

Ju

2023, 1, 5

For each bipartite graph G, there exists a graph $\tilde{G}$ such that $G \preceq \tilde{G}$ and $\tilde{G}$ is $\Delta(G)$-regular.

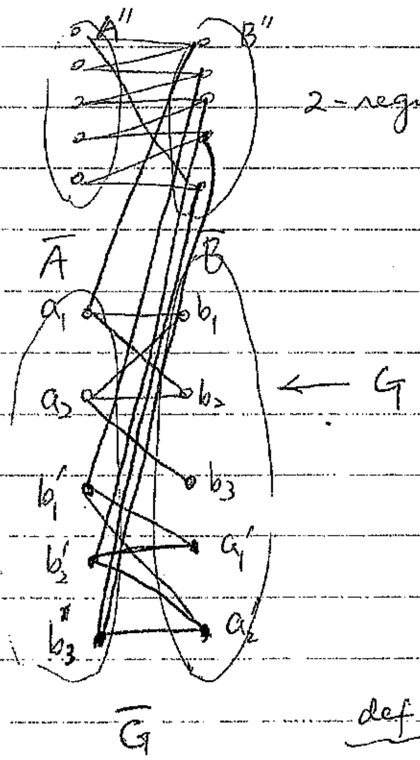
Proof. Let $G = (A, B)$ with $|A| \le |B|$.

Let $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$. $(m \le n)$

First, we construct a graph $\bar{G}$: $\underbrace{(A \cup B', B \cup A')}_{=(\bar{A}, \bar{B})}$ where

$A' = \{a_1', a_2', \cdots, a_m'\}$ and $B = \{b_1', b_2', \cdots, b_n'\}$ and $E(\bar{G}) = E(G) \cup \{\{b_j', a_i'\} \mid a_i b_j \in E(G)\}$. (See Figure for example) In fact, $\langle A \cup B \rangle_{\bar{G}} \cong \langle A' \cup B' \rangle_{\bar{G}}$. Hence, $\sum_{v \in A \cup B'} \deg_{\bar{G}}(v) = \sum_{v \in B \cup A'} \deg_{\bar{G}}(v) = 2\|G\|$, $|\bar{A}| = |\bar{B}|$, $\sum_{v \in A \cup B'} (\Delta(G) - \deg_{\bar{G}}(v)) = \sum_{v \in B \cup A'} (\Delta(G) - \deg_{\bar{G}}(v)) = \mathrm{def}(\bar{G})$.

Now, based on $\mathrm{def}(\bar{G})$, we have two cases to consider.



$\mathrm{def}(\bar{G}) = 5$

Case 1 $\mathrm{def}(\bar{G}) \ge \Delta(G)$

Construct a bipartite graph G'' such that $G'' = (A'', B'')$, $|A''| = |B''| = \mathrm{def}(\bar{G})$ and G'' is $(\Delta(G) - 1)$-regular. The proof then follows by connecting A'' to $\bar{B}$ and B'' to $\bar{A}$.

Case 2 $def(\overline{G}) < \Delta(G)$.

Let $|A''| = |B''| = \Delta(G)$, and $G'' \cong K_{\Delta(G),\Delta(G)} - M$, M is a matching of size $def(\overline{G})$. The proof follows by connecting the vertices in G'' (which are incident to M) and the vertices in $\overline{G}$ whose degrees are less than $\Delta(G)$.

So, the graph $\tilde{G}$ is defined on $(A'' \cup \overline{A}, B'' \cup \overline{B})$.

Moreover, $\tilde{G}$ is $\Delta(G)$-regular with partite set size $\max\{|A|+|B|+def(G), |A|+|B|+\Delta(G)\}$

Theorem 87

Let G be a planar graph with $\Delta(G) \geq 10$, then G is of Class 1.

Proof. We shall apply a lemma obtained by Vizing.

Vizing's adjacency lemma

First form: If G is of Class 2, then every vertex of G is adjacent to at least two major vertices. In particular, G contains at least three major vertices.

<u>Second Form</u>: Let G be a connected graph of Class 2 that is minimal with respect to edge coloring. <u>If $uv \in E(G)$ and $\deg_G(u) = m$, then v is adjacent to at least $\Delta(G) - m + 1$ major vertices.</u> (*)

We shall apply the 2nd form to prove the theorem.

Suppose not. Let G be a counterexample with minimum size. Thus, G is planar, $\overset{k}{\overset{\|}{\Delta(G)}} \geq 10$ and $\chi'(G) = k+1$. Clearly, G is minimal with respect to chromatic index. Since G is planar, G contains vertices of degree 5 or less, let S be the set of all such vertices. Define $H = G - S$. Again, H is planar, H contains a vertex w such that $\deg_H(w) \leq 5$. By the fact $\deg_G(w) > 5$, w is adjacent to some vertices of S. Let $vw \in E(G)$ where $v \in S$. In G, $\deg_G(v) \leq 5$. By (*), w is adjacent to at least $\Delta(G) - 5 + 1 \ (\geq 6)$ vertices (major) of degree $\Delta(G)$. This implies that w is adjacent to at least 6 vertices of H since all major vertices are in H. Hence, $\deg_H(w) \geq 6$. $\rightarrow\leftarrow$

This concludes the proof. ∎

(◦) $\sigma(G)$: Vertex cover number ; $\alpha_1(G)$: matching number

Theorem 88 (König–Egeváry) For bipartite graphs G, $\sigma(G) = \alpha_1(G)$.

Proof. We shall apply max-flow min-cut theorem to prove the theorem. First, we define a network as in Figure 1. Let $A = \{a_1, a_2, \cdots, a_m\}$, $B = \{b_1, b_2, \cdots, b_n\}$ and $G = (A, B)$. Then, the network is defined by letting u and v be source and sink respectively, $cap(u, a_i) = 1$ for $i = 1, 2, \cdots, m$, $cap(b_j, v) = 1$ for $j = 1, 2, \cdots, n$, and $cap(a_i, b_j) = |G| + 1$ if $a_i b_j$ is in $E(G)$. (Note that all arcs are from A to B.

Since $\sigma(G) \geq \alpha_1(G)$ as mentioned above, it suffices to show that $\alpha_1(G) \geq \sigma(G)$.

Now, let f be a maximum flow, It is easy to see that $val\, f = \alpha_1(G)$. This is due to the fact that all the arcs from u and into v are of capacity 1. (No two arcs can be out of two vertices in A and ended in a vertex of B.)

So, it is left to consider the minimum cut, let it be $K = (X, \bar{X})$ where $u \in X$, $v \in \bar{X}$, $A \cap \bar{X} = A'$ and $B \cap X = B'$, see Figure 2. Hence, K contains arcs from u to A', $A \setminus A'$ to $B \setminus B'$ and B' to v. Notice that $cap\, K \leq |G|$, for example, let $X = \{u\}$.

This implies that the following (*) is true and $A' \cup B'$ is a vertex cover and cap $K = |A'| + |B'| = \mathrm{val}\, f = \alpha_1(G)$. By the fact, $|A'| + |B'| \geq \alpha(G)$, we have $\alpha_1(G) \geq \alpha(G)$ ∎

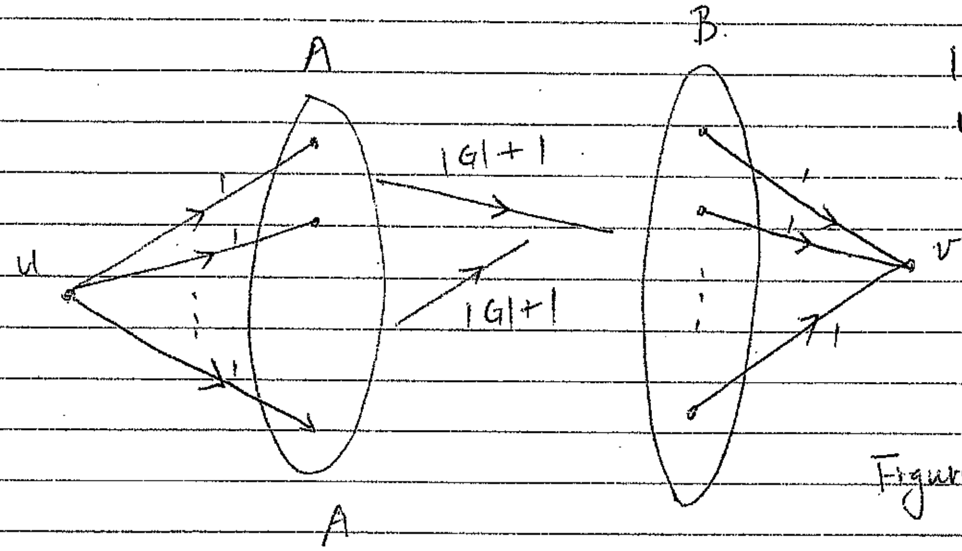


Figure 1. Network

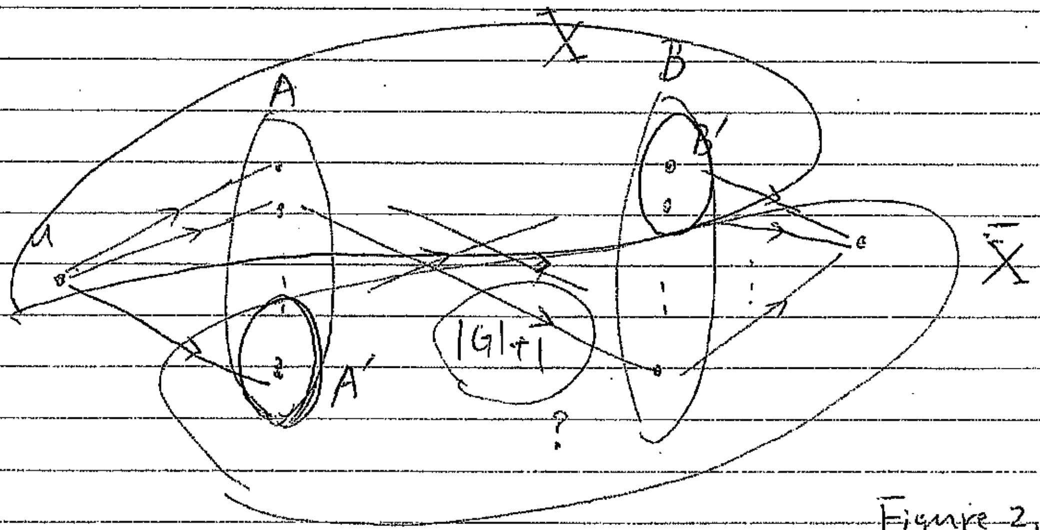


Figure 2. Cut

(*) In $(X, \bar{X})$, there exist no edges from $A \setminus A'$ to $B \setminus B'$. For otherwise, it is not a minimum cut. This implies that all edges are incident vertices in $A' \cup B'$. $A' \cup B'$ is a vertex cover.

Theorem 89

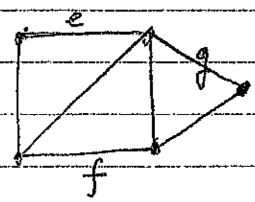
Let G be a graph of order p which has no isolated vertices. Then, $\alpha(G) + a(G) = p$.

Proof. Let S be a vertex cover with $a(G)$ vertices of G. Then, $V(G) \setminus S$ is an independent set. Hence, $|V(G) \setminus S| \leq \alpha(G)$ and thus $p - |S| \leq \alpha(G)$. This implies that $p \leq \alpha(G) + |S| = \alpha(G) + a(G)$. On the other hand, let T be an independent set of G such that $|T| = \alpha(G)$. Then, $G - T$ is vertex cover of G. By the fact $\underset{(\min.)}{a(G)} \leq |G - T| = p - \alpha(G)$, we have $p \geq a(G) + \alpha(G)$. ∎

Definition (Edge-cover)

An edge cover of a graph is a set of edges M such that all vertices of G are incident to M, i.e., for each $v \in V(G)$, v is incident to an edge in M.

e.g.



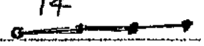
$\{e, f, g\}$ is an edge cover.

The edge cover number of G, $\sigma_1(G) = \min\{|M| \mid M \text{ is an edge cover}\}$

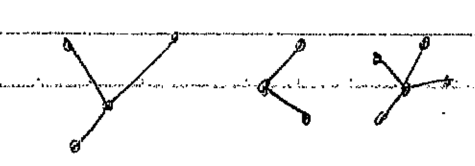
(•) $\alpha_1(G) \geq \lceil \frac{|G|}{2} \rceil$. (Each edge can cover two vertices.)

Theorem 90 $\alpha_1(G) + \alpha_1(G) = p$. (G is a connected graph.)

Proof. Let M be a matching in G with $\alpha_1(G)$ edges. Then, for each vertex not in M, v, is incident to a vertex in M if v is in an edge of G. Assume that there t vertices not in M, i.e, $p = 2|M| + t$. Now, by taking every edge in the matching M and the set of t edges not in M but incident to M, we have an edge cover with $|M|+t$ edges. This implies that $\alpha_1(G) \leq |M|+t$. As a consequence, we have $p = 2|M|+t = \alpha_1(G) + |M|+t \geq \alpha_1(G)+\alpha_1(G)$.

On the other hand, let N be an edge cover of G with minimum number of edges, i.e, $|N| = \alpha_1(G)$. Notice that $\langle N \rangle_G$ is a disjoint union of stars. (You can not find P_4  in $\langle N \rangle_G$.) Assume that there are t stars. Then, $p = |N| + t$. By the fact that in $\langle N \rangle_G$ we can find a matching of size t, $p \leq |N| + \alpha_1(G)$. ($\alpha_1(G) \geq t$.)

maximum mat number

( $|N| = 9$, $t=3$, $p=12$)

Theorem 90'

Let G be a graph with $V(G)=\{v_1, v_2, \cdots, v_p\}$. Then, we can use the vertices of G as variables to obtain a generating function for all dominating sets of G.

Proof. Let f be defined as follows:

$$f(v_1, v_2, \cdots, v_p) = \prod_{i=1}^{p} \left(v_i + \sum_{u \in N_G(v_i)} u\right).$$

Then, each summand is a product $v_1^{c_1} v_2^{c_2} \cdots v_p^{c_p}$ where $0 \le c_j \le p$. Now, let $S = \{v_j \mid c_j > 0,\ j=1,2,\cdots,p\}$. If $u \in V(G) \setminus S$, say $u = v_k$, then in the product $v_1^{c_1} v_2^{c_2} \cdots v_p^{c_p}$, $c_k = 0$. But, one of its neighbor has been selected. This implies that v_k is incident to a vertex of S. ▨

(•) For small order graphs, this is a good way to find dominating sets. In fact, the term with maximum of "0" in powers provide a dominating set with minimum size and thus the domination number is determined.