Theorem 91

Let G be a (p, q)-graph and $A(G) = A$. Then,

(1) the number of triangles in G is $\frac{1}{6}\,tr(A^3)$;

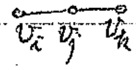
(2) the number of 4-cycles in G is $\frac{1}{8}\left[tr(A^4) - 2q - \sum_{i \neq j} a_{i,j}^{(2)}\right]$

where $a_{i,j}^{(2)}$ is the (i,j)-entry in A^2; and

(3) the number of 5-cycles in G is $\frac{1}{10}\left[tr(A^5) - 5\,tr(A^3) - 5 \sum_{i=1}^{p} \sum_{j=1}^{p} (a_{i,j} - 2) \cdot a_{i,i}^{(3)}\right]$.

Proof. It follows from the fact that the number of walks of length k from v_i to v_j is equal to $A^{(k)}(i,j)$. This can be proved by induction on k.

Hence, if triangles are concerned, then we consider $A^{(3)}(i,i)$, i.e., $tr(A^3)$. Since for each triangle (v_i, v_j, v_k), there are 6 different ways of 3-walks: $\langle v_i, v_j, v_k, v_i \rangle$, $\langle v_i, v_k, v_j, v_i \rangle$, $\langle v_j, v_k, v_i, v_j \rangle$, $\langle v_j, v_i, v_k, v_j \rangle$, $\langle v_k, v_i, v_j, v_k \rangle$ and $\langle v_k, v_j, v_i, v_k \rangle$, the result follows by using $\frac{1}{6}\,tr(A^3)$.

For 4-cycles and 5-cycles, we have to take away those 4-walks (and 5-walks) which are not for cycles, for example  $v_i\ v_j\ v_k$ $\Rightarrow \langle v_i, v_j, v_k, v_j, v_i \rangle$.

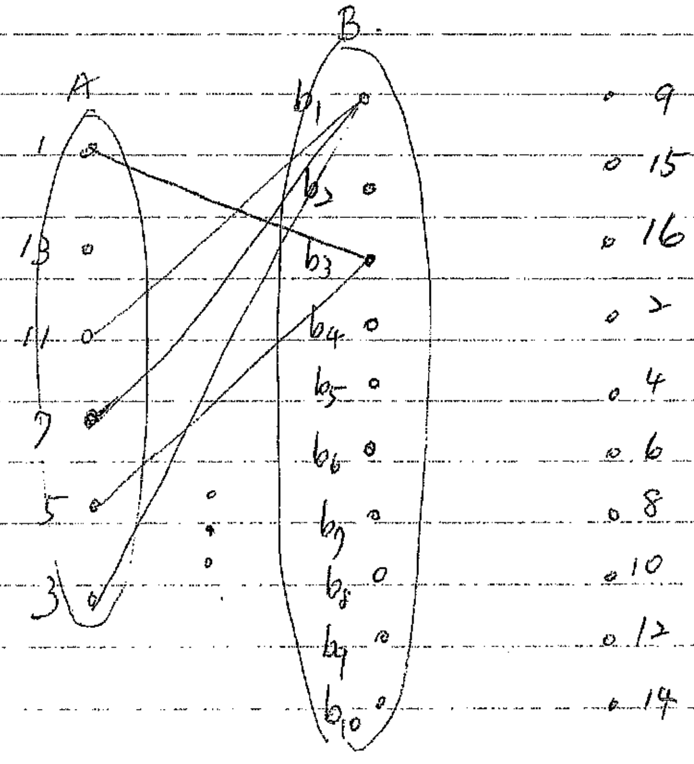
Check (2) and (3) yourself. ∎

Theorem 9.2

Let $G=(A,B)$ be a tree of order at most 16. Then, G has a prime labeling.

Proof. (Sketched) We present a proof of the case $|A|=6$ and $|B|=10$. First, we label A by using $S=\{1,13,11,7,5,3\}$. Then, the labeling of the vertices can be found from the set $\underbrace{\{1,2,\cdots,16\}}_{[16]}\setminus S$. Now, denote $B=\{b_1,b_2,\cdots,b_{10}\}$. We claim that there exists a labeling of the vertices of B by using $[16]\setminus S$.

Let $B_i=\left\{x\in[16]\setminus S \;\middle|\; \begin{array}{l}\gcd(x,s)=1\\ \text{for each}\\ x\in S\cap N_G(b_i)\end{array}\right\}$. Now, consider $\underbrace{\left|\bigcup_{j=1}^{k} B_j\right|}_{|S_k|}$. Clearly, if $k=1$, then $|S_k|\geq 1$ since $\gcd(2,s)=1$ for each $s\in S$. In fact,



each B_i contains at least 4 elements, 2, 4, 8 and 16. By the fact, any two B_i's have at most one common neighbor, we can verify that $|S_k|\geq k$ for $6\leq k\leq 10$. Hence, By Hall's condition, the proof follows. ∎

Theorem 93 (Alon, 1990)

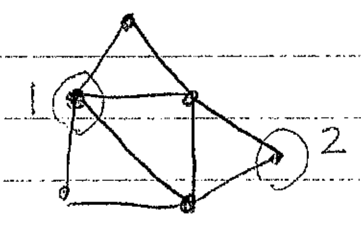
Let G be a graph of order p. Then, $\gamma(G)$ (the domination number) of G, $\gamma(G) \leq n[1+\ln(\delta(G)+1)]/(\delta(G)+1)$.

Proof. (Probabilistic Method)

Let S be a subset of $V(G)$ with the probability of each vertex $p =_{def} \frac{\ln(\delta(G)+1)}{\delta(G)+1}$. Let $T = \{x \mid x \notin S, N_G(x) \cap S = \phi\}$. Since for each $y \notin S \cup T$, $N_G(y) \cap S \neq \phi$, $S \cup T$ is a dominating set of G. By the expectation of $E(S \cup T) = E(S) + E(T) \leq np + n \cdot (1-p)^{\delta(G)+1}$ $\leq np + n \cdot e^{-p(\delta(G)+1)} = n(p + \frac{1}{\delta(G)+1})$. This implies that there exists a dominating of size at most $n \cdot [1+\ln(\delta(G)+1)]/(\delta(G)+1)$. ∎

(o) A greedy algorithm for finding the dominating set:

Choose the vertices of a dominating set one by one following the idea: A vertex that covers the maximum number of vertices which are not cover yet is pick.



Date p is not a constant.

Theorem 94 (Omit the proof)

Let $p = p(n)$. Then, we have

(1) $p(n) = n^{-2}$ $\longrightarrow$ No edges.

(2) $p(n) = n^{-\frac{3}{2}}$ $\longrightarrow$ G has a nontrivial component which grows like a tree.

(3) $p(n) = n^{-1}$ $\longrightarrow$ Contain a cycle.

(4) $p(n) = \frac{\ln n}{n}$ $\longrightarrow$ Connected.

(5) $p(n) = (1+\varepsilon) \cdot \frac{\ln n}{n}$ $\longrightarrow$ Contains a Hamilton cycle.

(•) The growth rate is getting smaller.

Theorem 95 (More about eigenvalues of $A(G)$)

Let G be a connected graph of order p and A be its adjacency matrix. Then, we have the following basic properties.

(1) For each eigenvalue λ, $|\lambda| \le \Delta(G)$.

(2) $\Delta(G)$ is an eigenvalue of G if and only if G is regular.

Moreover, if $\Delta(G)$ is an eigenvalue of G, then the multiplicity of $\Delta(G)$ is 1.

(3) If $-\Delta(G)$ is an eigenvalue of G, then G is regular and bipartite.

(4) If G is bipartite and λ is an eigenvalue then $-\lambda$ is also an eigenvalue, moreove, they have the same multiplicity.

(5) The maximal eigenvalue $\lambda_{max}(G)$ satisfies $\delta(G) \leq \lambda_{max}(G) \leq \Delta(G)$.

(6) If $H \trianglelefteq G$, then $\lambda_{min}(G) \leq \lambda_{min}(H) \leq \lambda_{max}(H) \leq \lambda_{max}(G)$.

Proof.

(1) Let $\vec{x}$ be an eigenvector with eigenvalue λ, i.e., $A\vec{x} = \lambda\vec{x}$.

Let $\vec{x} = (x_1, x_2, \cdots, x_p)$ and $|x_i| \leq 1$ (by re-scaling $\vec{x}$). Suppose that $|x_j| \geq |x_i|$ for each $i = 1, 2, \cdots, p$. For convenience, let $x_j = 1$.

($|x_j|$最大，則令 $\vec{x} \leftarrow \frac{\vec{x}}{\pm|x_j|}$.)

Then, $|\lambda| = |\lambda \cdot x_j|$

$$= \left|\sum_{i=1}^{p} a_{j,i} \cdot x_i\right| \leq \sum_{i=1}^{p} a_{j,i} |x_i| \leq |x_j| \cdot \deg_G(v_j) \leq \Delta(G). \quad ■$$

(2) If $\Delta(G)$ is an eigenvalue, then as in (1), let $|x_j| = 1$, and we have $\Delta = \Delta x_j = \sum_{i=1}^{p} a_{j,i} x_i$. Hence $x_i = x_j = 1$ and $\deg_G(v_j) = \Delta$ whenever $v_i \sim_G v_j$. Therefore, $\deg_G(x_i) = \Delta$. Now, by the same argument, if $v_k \sim v_i$, $\deg_G(v_k) = \Delta$, then G is Δ-regular by

the fact that G is connected. This also implies that the eigenvector is $\vec{1} = (1, 1, \cdots, 1)$. The reverse statement is easy to see.

(3) If $-\Delta(G)$ is an eigenvalue, then by (2) we have $\deg_G(v_j) = \Delta$ and $x_i = -x_j = -1$ whenever $v_i \sim_G v_j$. Since two vertices are adjacent if they have distinct weights (x_i and x_j) 1 and -1, the vertex set of G can be partitioned into two subsets V_1 and V_2 such that $v_i \sim_G v_j$ iff their corresponding weights are different (1 or -1). Hence, G is bipartite.

(4) It follows by considering $Ker(A - \lambda I_p)$ and $Ker(A + \lambda I_p)$.

Let $G = (V_1, V_2)$ and $\vec{b} = (b_1, b_2, \cdots, b_p)$ such that $b_i = 1$ if $v_i \in V_1$ and $b_i = -1$ if $v_i \in V_2$. Now, if $A\vec{x} = \lambda\vec{x}$ and $v_i \in V_1$ then

$$A \cdot (\underbrace{\vec{b} \otimes \vec{x}}_{\overset{\text{def}}{=} (b_1x_1, b_2x_2, \cdots, b_px_p)})_i = \sum_{j=1}^{p} a_{ij} b_j x_j = \underbrace{\sum_{v_j \in V_1} a_{ij} x_j}_{= 0 \ (v_i \in V_1)} - \sum_{v_j \in V_2} a_{ij} x_j$$

$$= -\sum_{v_j \in V_1} a_{ij} x_j - \sum_{v_j \in V_2} a_{ij} x_j = -\sum_{j=1}^{p} a_{ij} x_j = -\lambda x_i = -\lambda (\vec{b} \otimes \vec{x})_i$$

This implies λ and $-\lambda$ occur the same number of times in solving $A\vec{x} = \lambda\vec{x}$, i.e., $m(\lambda) = m(-\lambda)$.

$m(\lambda)$: multiplicity of λ

(5) By (1), we have $\lambda_{max}(G) \leq \Delta(G)$. Now, we claim the other inequality. Let the numerical range of A be $V(A)$, i.e., $V(A) = \{ \langle A\vec{x}, \vec{x} \rangle = \vec{x}^{t} A \vec{x} \mid |\vec{x}| = 1 \}$. Hence, let $\vec{1} = (1,1,\cdots,1)$, and we have $\frac{1}{p}\langle A\vec{1}, \vec{1} \rangle \in V(A)$.

Now, $\lambda_{max} = \max V(A) \geq \frac{1}{p}\langle A\vec{1}, \vec{1}\rangle = \frac{1}{p}\sum_{k=1}^{p} \deg_G(v_k) \geq \delta(G)$. ▨

(6) Let H be an induced subgraph of order $p-1$, i.e., $H = \langle \{v_1, v_2, \cdots, v_p\} \rangle_G$

Then, $\lambda_{max}(H) = \langle A'\vec{y}, \vec{y} \rangle$ where $A' = A(H)$ and $\langle \vec{y}, \vec{y} \rangle = 1$.

Now, consider $\vec{x} = (y_1, y_2, \cdots, y_{p-1}, 0)$ where $\vec{y} = (y_1, y_2, \cdots, y_{p-1})$.

Clearly, $\langle A\vec{x}, \vec{x} \rangle = \underbrace{\langle A'\vec{y}, \vec{y} \rangle}_{=\lambda_{max}(H)}$ and $\langle \vec{x}, \vec{x} \rangle = 1$. Since $\langle A\vec{x}, \vec{x} \rangle \in V(A)$, i.e., $\lambda_{max}(H) \in V(H)$. This implies that $\lambda_{max}(G) \geq \lambda_{max}(H)$. The other inequalities can be shown similarly. ▨

(∘) A graph G has an $\mathcal{H}$-decomposition if $E(G)$ can be partitioned into subsets $E_1, E_2, \cdots, E_k$ such that for each $i = 1, 2, \cdots, k$, $\langle E_i \rangle_G \in \mathcal{H}$.

(∘) If $\mathcal{H} = \{H\}$, then an $\mathcal{H}$-decomposition of G can be referred as an H-decomposition of G.

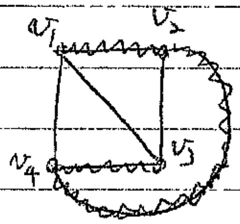
(∘) A graph G has an $\mathcal{H}$-packing if $E(G)$ contains edge-disjoint subsets such that each of them induces a graph in $\mathcal{H}$. An H-packing of G can be defined accordingly.

(∘) A graph G has an $\mathcal{H}$-covering if $E(G)$ is a subset of a edge-disjoint union of graphs in $\mathcal{H}$. An H-covering of G can be defined as well.

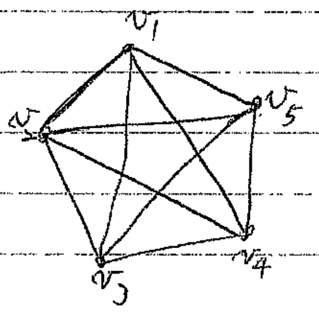
(∘∘) In packing, the edges not used induce a subgraph which is known as the "leave" of the packing. Similarly in covering, the extra edges used induce a padding of the covering.

(∘) If the graph G is the complete graph K_n, then the H-decomposition, H-packing and H-covering is also referred to as that of order n.

Examples



$\Rightarrow$ P_4-decomposition $\{\langle v_1, v_2, v_4, v_3\rangle, \langle v_2, v_3, v_1, v_4\rangle\}$



$\Rightarrow$ C_3-packing: $\{(v_1, v_4, v_5), (v_1, v_2, v_3)\}$ with leave a C_4, (v_2, v_3, v_4, v_5).

C_3-covering: $\{(v_1, v_4, v_5), (v_1, v_2, v_3), (v_2, v_4, v_4), (v_2, v_4, v_5)\}$ with a padding .

Theorem 96

For each odd integer $n \geq 3$, K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles. For each even integer n, K_n can be decomposed into $\frac{n}{2}$ Hamilton paths.

Proof. The following construction is known as the Waleki's method.

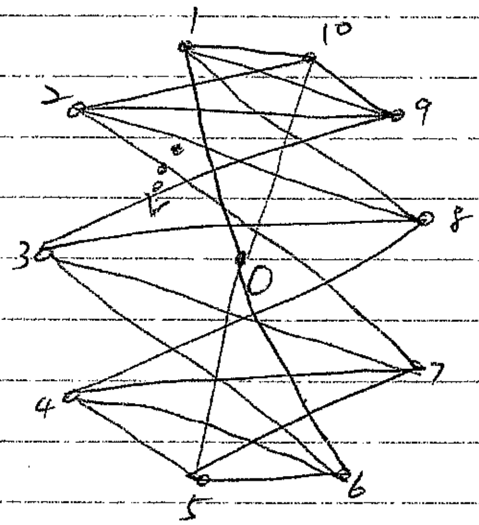
Let $V(K_n) = \mathbb{Z}_n$ and the cycles are :

$$(0, 1, n-1, 2, n-2, \cdots, \tfrac{n-1}{2}, \tfrac{n+1}{2}),$$

$$(0, 2, 1, 3, n-1, \cdots, \tfrac{n+1}{2}, \tfrac{n+3}{2}),$$

$$\vdots$$

$$(0, \tfrac{n-1}{2}, \tfrac{n-3}{2}, \tfrac{n+1}{2}, \tfrac{n-5}{2}, \cdots, n-2, n-1). \quad \blacksquare$$



By deleting a vertex in K_{n+1} (n even), we obtain the decompositio of K_n into $\frac{n}{2}$ Hamilton paths. $\blacksquare$

(*) This theorem has been extended to cycles and paths with prescribed length. We list the theorem and omit their proofs. (The cycle case is very complicate.)

(many authors together)

<u>Theorem 96′</u> (Alspach et. al, 2001)

For each odd integer larger than 3 and an integer $3 \leq m \leq n$, the complete graph K_n (n is odd) and $K_n - I$ (n is even) can be decomposed into m-cycles provided $m \mid \binom{n}{2}$ (for odd n) and $m \mid \binom{n}{2} - \frac{n}{2}$ (for even n) respectively.

(◦) The case $m = 3$ was proved in 1847 by T.P. Kirkman, and the case $m = n$ mentioned above was obtained long time ago.

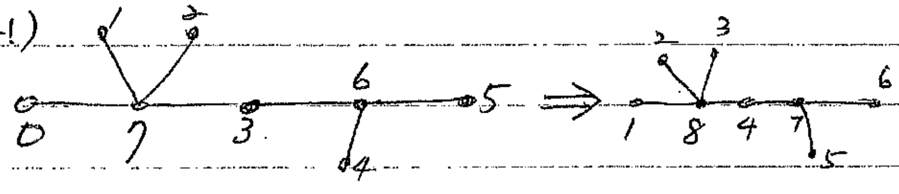
(*) An important tool for decomposition.

<u>Definition</u> (Graceful labeling, β-labeling)

A graceful labeling of a graph G is a 1-1 mapping $f : V(G) \longrightarrow \{0, 1, 2, \cdots, \underline{\max\{\|G\|, |G|-1\}}\}$ such that the weights of edges uv defined by $|f(u) - f(v)|$ are all distinct.

If G is connected, this <u>value</u> takes $\|G\|$.

<u>Example</u> (Shown earlier!)

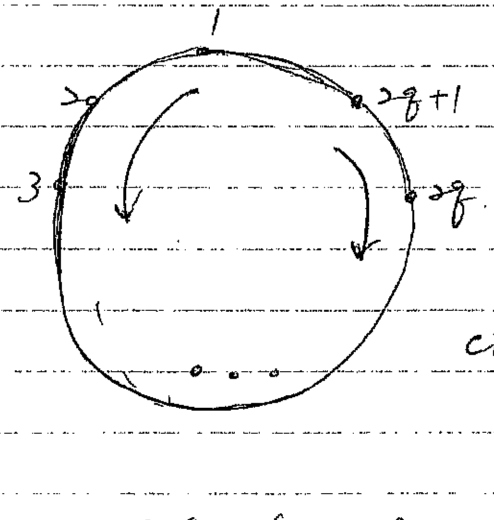


Theorem 97

Let G be a graph of size q and G has a graceful labeling. Then, K_{2q+1} can be decomposed into $2q+1$ copies of G.

Proof. Let $V(K_{2q+1}) = \mathbb{Z}_{2q+1}$. By arranging the vertices on a cycle, see Figure below, we notice that any two vertices have a circular distance at most q. More precisely, $dist(i, j)$ (for $j > i$) $= \min\{ j-i, (2q+1)-(j-i)\}$.

Now, we can add the labels of G for each one of them (taking modulo $2q+1$) and obtain the desired decomposition.

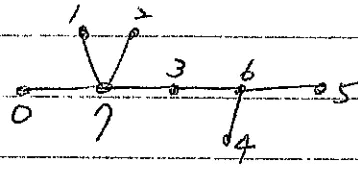


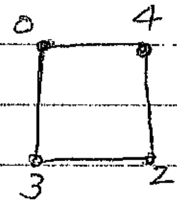
cicular distance

(*) If G has a graceful labeling f and the labeling has an extra property such that $\exists\, c_f \in \mathbb{R}$ satisfying for each uv, either $f(u) \geq c_f > f(v)$ or $f(v) \geq c_f > f(u)$, then G has an α-labeling.

The idea of Theorem 97 is known as "difference method" and G with labeling is a "base graph".

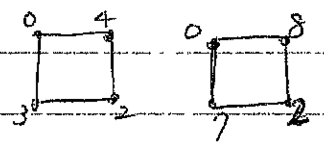
Example

The labeling of  is an α-labeling since we can choose $c_f = 3$ or 4 or 3.5. The following labeling is also an α-labeling, $c_f = 2.5$.



$\Rightarrow K_9 \xrightarrow{} C_4$'s

$\Rightarrow K_{17} \xrightarrow{} C_4$'s



Theorem 98

If G has an α-labeling and $\|G\| = q$, then $G \mid K_{2tq+1}$ where $t \in \mathbb{N}$. ($G \mid K_{2tq+1}$ denotes K_{2tq+1} has a G-decomposition.)

Proof. By Theorem 97, $G \mid K_{2q+1}$ can be obtained by a graceful labeling of G. Now, if G has an α-labeling, we may change the labels to find t base graphs for the decomposition of K_{2tq+1}. As mentioned above on the case of C_4's, for each label larger than c_f, we add $q, 2q, \cdots, (t-1)q$ respectively. This gives a collection of t base graphs (with labels). By difference method, we have the proof. (All differences from 1 to tq have been used exactly once) ∎

(*) If G has an α-labeling, then G must be a bipartite graph. The two partite sets of G are obtained by using the labels, larger than g and smaller than g respectively.

(•) A graph G may have β-labeling but not α-labeling.

(**) One of the most beautiful conjectures on labelings is

<u>Graceful Tree Conjecture</u>: Every tree has a graceful labeling.

Of course, you may also conjecture that every tree has an α-labeling (but this is in general not true).

(•) We have shown that for each graph G there exists a $\Delta(G)$-regular graph H such that $G \preceq H$. In fact, we can say more about this type of supergraph.

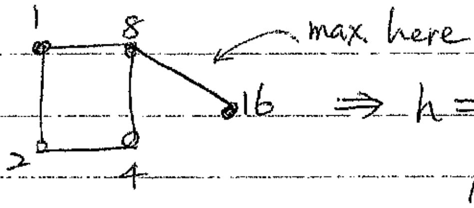
<u>Theorem 99</u>

Let G be a graph of size q without isolated vertices. Then, there exists a regular graph H such that $G | H$. More precisely, H is a $2q$-regular graph.

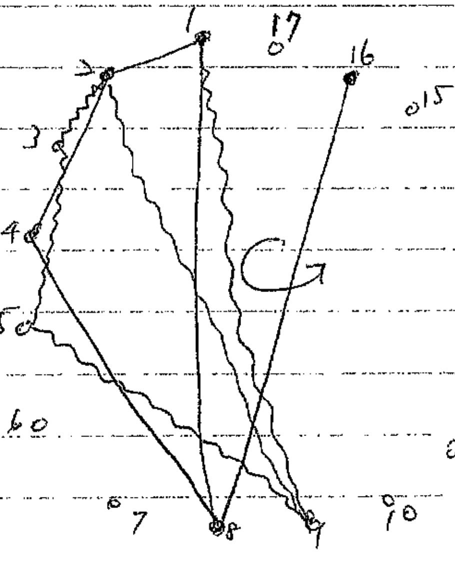
Proof. Let $G=\{v_1, v_2, \cdots, v_p\}$ and f is a labeling of G such that $f(v_i)=2^{i-1}$. Then, all edges will receive distinct weights $|f(u)-f(v)|$. We shall construct a $2q$-regular graph of order h such that $h = 1+2\cdot\max\{|f(v_i)-f(v_j)| \mid v_iv_j \in E(G)\}$.

Let $V(H)=\mathbb{Z}_h$ and G be the graph with its vertices the labels from f, see the following figure for an example. Then, by difference method, the base graph will generate a regular graph which use the weights in G exactly once. Since $\|G\|=q$, the graph obtained H will be a $2q$-regular graph of order h. ■

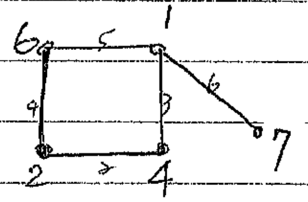
(*) We can decrease the order of H by giving another labeling satisfying all $|f(u)-f(v)|$'s are different for $uv\in E(G)$.

G: 

$\Rightarrow h=17$



Another example



H: 10-regular graph of order 13.

(•) "Graph Decomposition" is one of the most important topic in Graph Theory, many results are also related to the study of Combinatorial Designs.

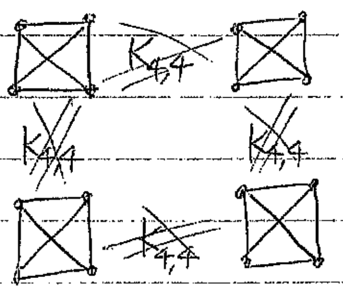
(*) Problem For which graph G, $K_3 | G$?

(*) Problem For which graph G, $C_4 | G$?

Theorem There exists a graph G with $\delta(G) < \frac{3}{4}|G|$ such that $K_3 \nmid G$.

Proof. For general n, the construction is similar to the following graph of order 16.

G:



$\Delta(G) = 11 < 12$

Since their are four bipartite subgraphs $K_{n,n}$ in G, the K_3-decomposition needs to use up all these edges by using one from K_n and two from $K_{n,n}$. ($K_{n,n}$ contains no odd cycles!) Hence, we need at least $\frac{1}{2}(4n^2)$ edges from four K_n's. But, $4K_n$ has $2n(n-1)$ edges which are not enough! The K_3-decomposition of G is not possible. ▨

<u>Nash-Williams Conjecture</u>

For any graph G of order p, G has a K_3-decomposition provided $\delta(G) \geq \frac{3}{4}p$ and $3 \mid \|G\|$.

How about C_4-decomposition? Keep moving forward!