Lecture 13

Graph Labelings

Vertex-labeling

$\varphi : V(G) \longrightarrow L$ (label-set), s.t. ...

Edge-labeling

$\psi : E(G) \longrightarrow L'$ (label-set), s.t. ...

Face-labeling

$\pi : F(G) \longrightarrow L''$ (label-set), s.t. ...

...-labeling

Combination of above labelings

Well-known labelings

① Colorings

② Magic or Anti-magic labelings (On edges)

③ (a, b)-labelings (On vertices)

④ Prime labeling (Vertex)

⑤ Graceful labeling, α-labeling, β-labeling, ρ-labeling, ...

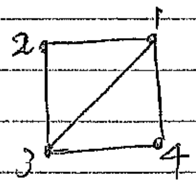
⑥ Sign labeling (Edges)

Lecture 13 Dec. 21, 22.

(∘) Prime Labeling

A prime labeling of a graph G is a mapping $f: V(G) \xrightarrow{1-1} \{1, 2, \cdots\}$ such that if $uv \in E(G)$, then $f(u)$ and $f(v)$ are relatively prime, i.e., $\gcd(f(u), f(v)) = 1$.

Example



(∘) $P(a,b)$: The set of primes between integers a and b.

$P(a,b]$, $P[a,b]$ and $P[a,b)$ can be defined accordingly.

(*) Bertrand's postulate: (Bertrand–Chebyshev Theorem)

For each $n \geq 2$, $P(n, 2n) \neq \phi$.

(∘) $\alpha(G)$: Independence number of G.

$\sigma(G)$: Vertex cover number of G

(S is a vertex cover of G if every edge e of $E(G)$ is incident at least one vertex of S.) $\Rightarrow V(G) \setminus S$ is an independent set.

Theorem 26

Let G be a graph of order n. Then the followings hold.

(1) If $\alpha(G) < \lfloor \frac{n}{2} \rfloor$, then G has no prime labelings.

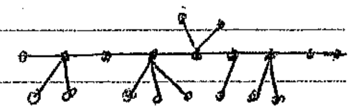
(2) If S is a vertex cover of G and $|S| \leq |P(n, 2n)| + 1$, then G has a prime labeling.

Proof. (1) follows from the fact that in all prime labelings (if exist), the set of vertices with even lebels induces an independent set. Hence, the graph must contain an independent set of size $\lfloor \frac{n}{2} \rfloor$.

(2) Use the labels 1 and $x \in P(n, 2n)$ to label the vertices in S, we obtain a prime labeling of G. (The vertices in $G-S$ can be labeled arbitrarily. ■

Use the facts above, we can easily find a prime labeling of a tree of order at most 10. The proof follows by letting $T = (A, B)$ and consider the cases $|A| = 1, 2, 3, 4, 5$.

Prime labeling Conjecture

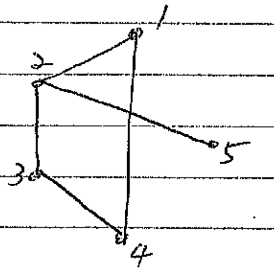


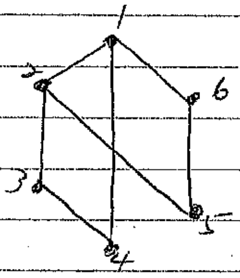
Every tree has a prime labeling.

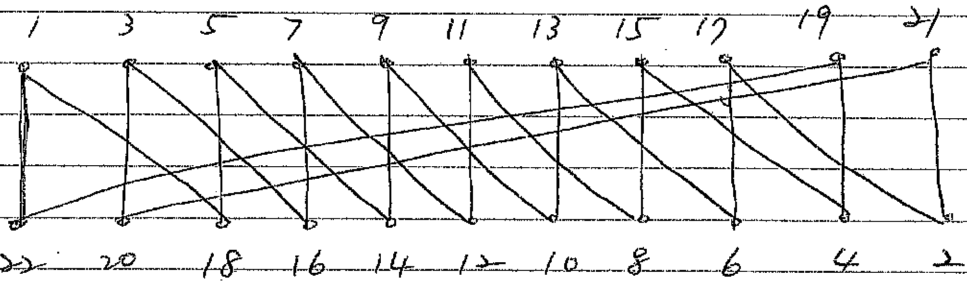
(*) The first goal should be the answer for caterpillar.

Prime sum graph

(i) The prime sum graph of order n, G_n, is defined on $[n]$ and two vertices i and j in $[n]$ are adjacent if $i+j$ is a prime.

G_5:


G_6:


G_{22}:


A part of edges, 23, 19, 41 are primes used for them.

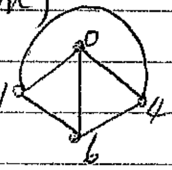
(1, 22, 19, 4, 15, 8, 11, 12, 7, 16, 3, 20, 21, 2, 17, 6, 13, 10, 9, 14, 5, 18) ← Hamilton cycle.

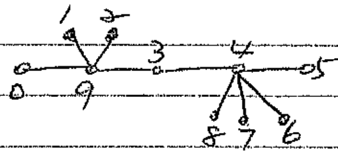
(ii) So, we are interested in determining "for which n, G_n has a Hamilton cycle?". Clearly, n must be even. (?)

(*) It is not difficult to check, $\underbrace{\text{for small } n}_{\geq 6}$, G_n is indeed hamiltonia

(0) <u>Graceful Labelings</u> (with $|G| \le \|G\|+1$)

A graceful labeling of a graph G is a mapping $f: V(G) \xrightarrow{1-1} \{0,1,2,\cdots,\|G\|\}$ such that the weights of edges uv, defined by $|f(u)-f(v)|$, are all distinct. In case that $|G| > \|G\|+1$, we use the mapping $f: V(G) \xrightarrow{1-1} \{0,1,2,\cdots,|G|\}$ instead. (Vertex version)

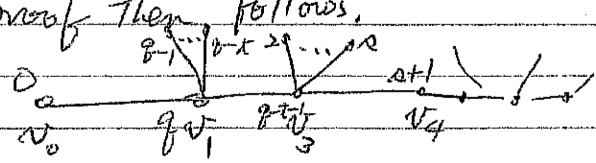




<u>Theorem 27</u>

Any caterpillar has a graceful labeling.

<u>Proof.</u> Starting from the end vertex from one side, label the vertex v_0 with 0, then the vertex v_1 incident to v_0 is labeled with $\|G\| = q$. Now, the neighbors of v_1 are labeled by $2, 3, \cdots$ for pendant vertices and let the largest label be used in v_3. Again, we shall start from the use of $q-1, q-2, \cdots$ and let the smallest label for v_4 to use. The proof then follows. ■



(*) There are special trees which have a graceful labeling. But, to prove that all trees do have graceful labelings remains unsolved.

<u>Graceful Tree Conjecture</u> (Ringel-Kotzig)

<u>All trees are graceful.</u>

(*) We remark here, for forests, we can also find graceful labelings.

<u>Theorem 77'</u>

Each matching of size n, M_n, has a graceful labeling.
(<u>Vertex version</u>)

<u>Proof.</u> This is a direct consequence of using Skolem sequences of order n. (See a couple of examples below.) ■

2 3 7 9 1 4 8 12 5 10 0 6 $n=6$

11 12 13 15 1 4 2 6 5 10 3 9 7 14 0 8 $n=8$

(**) There are many problems in graph labelings, you may refer to the following reference for more informations.

A dynamic survey of graph labeling by J. A. Gallian.
(502 pages!)

Probabilistic Method and Random Graphs

(*) The notion of random graphs is different from probabilistic method. Here are two theorems which uses probabilistic method.
(Find the lower bound of $R(s)$ is another example)

Theorem 28

There exists a tournament T_n such that T_n has at least $\lfloor n! / 2^{n-1} \rfloor$ directed Hamilton paths. (The proof of existence is not difficult)

Proof. In K_n, there are $n!$ Hamilton paths. (Starting from one of the vertices of K_n and choose one edge at a time without passing vertices which have been selected.) Now, if each edge is assigned with an orientation, the probability for any one Hamilton path to be a directed Hamilton path is $\frac{1}{2^{n-1}}$. (each edge has $\frac{1}{2}$ chance to be in the right direction.) So, if X is the random variable for the number of directed Hamilton paths, $E(X)$ (Expectation) $= n! \cdot \frac{1}{2^{n-1}}$ and therefore, there exists a tournament satisfying this value. ■

Theorem 29

The independence number of G, $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1+\deg_G(v)}$.

<u>Proof.</u> Let f be a random labeling of G by using $1, 2, \cdots, |G|$. For convenience, let $V(G) = \{v_1, v_2, \cdots, v_p\}$ and f is 1-1 mapping from $V(G)$ onto $\{1, 2, \cdots, p\}$. Now, for each v, there exists a unique $u \in N_G[v]$ such that $f(u) = \min.\{f(x) \mid x \in N_G[v]\}$. Now, let S be the set of vertices v in $V(G)$ such that $f(v) = \min.\{f(x) \mid x \in N_G[v]\}$. That's if v has the smallest label, then put v in S.

Now, clearly S is an independent set. (?) Moreover, the probability of being the smallest label among all its neighbors is $\frac{1}{\deg_G(v)+1}$, we conclude the proof since $\alpha(G) \geq |S| = \sum_{v \in V(G)} \frac{1}{1+\deg_G(v)}$. ■

<u>Random Graphs</u>

There are different models, here we consider one of the most popular one.

<u>Model A</u> $G(n, p)$, $0 \leq p \leq 1$.

The probability of the existence of an edge (independently) is p and the graph induced by using existent edges is G_p.

(∘) We use G^n to denote the distribution of graphs of order n.
Let q_n be the probability of the existence of "property" Q when the graphs considered are of order n.

(∘∘) If $\lim_{n\to\infty} q_n = 1$, then we say "Q" almost always holds. In this case, we say almost all graphs have property "Q".

Theorem 80 (Gilbert, 1959)
Let p be a constant such that $0 < p \le 1$. Then, almost all graphs G_p are connected.

Proof. Suppose not. Then, there are graphs G (of order n) which is not connected. Hence, there exists a proper subset $S \subseteq V(G)$ such that $\langle S, V(G)\setminus S\rangle$ contains no edges. This implies that the probability q_n of the existence of disconnected graphs of order n satisfies

$$0 \le q_n \le \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (1-p)^{k\cdot(n-k)} \cdot p^c \quad (c \text{ is a constant}) \ \# \text{ of edges}$$

$$\le \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} n^k (1-p)^{k(n-k)} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(n\cdot(1-p)^{n-k}\right)^k$$

$$< \frac{y}{1-y} \ (\text{where } y = (n\cdot(1-p)^{n-k})).$$

Since $\lim_{n\to\infty} n(1-p)^{n-k} = 0$, $\lim_{n\to\infty} \frac{y}{1-y} = 0$. Hence, $\lim_{n\to\infty} q_n = 0$ and thus almost all graphs are connected. ∎

(c) It is not difficult to see the connectedness should be quite strong, not only 1-connected. We can infact claim that any cut set of size $k-1$ is not available for a fixed k.

Following the arguments from Theorem 80, we are able to show, if n is large enough, then a random graph obtained from constant p will give a graph with high connectedness.

Theorem 81. For every constant $p \in (0,1)$ and $k \in \mathbb{N}$, almost all graphs are k-connected.

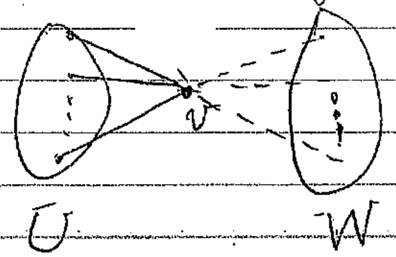
Proof. First, we claim G_p has property "$P_{i,j}$". ($P_{i,j}$ is the property that for any disjoint vertex sets U and W with $|U| \le i$ and $|W| \le j$, there exists a vertex $v \notin U \cup W$ such that $U = N_G(v)$ but $W \cap N_G(v) = \phi$. (See Figure below) Let $q = (1-p)$. Then, the probability of such v is $p^{|U|} \cdot q^{|W|} \ge p^i \cdot q^j$. ( )

Hence, the probability of no such v exists is

$$(1-p^{|U|} \cdot q^{|W|})^{n-|U|-|W|} \le (1-p^i \cdot q^j)^{n-i-j} \quad \text{for } n \ge i+j.$$

Now, there are at most n^{i+j} $\langle U, W \rangle$ pairs, and thus the probability of "$\sim P_{i,j}$" is at most $n^{i+j} \cdot (1-p^i q^j)^{n-i-j}$. By the fact i and j are constants, we have the probability "0" when $n \to \infty$.

not $P_{i,j}$



To prove the theorem, let $i=2$ and $j=k-1$. Since almost all graphs G have property $P_{2,k-1}$. Let W be an arbitrary set of $k-1$ vertices. Then, for any two vertices $x, y \in V(G)\setminus W$, then either x is adjacent to y or x and y have a common neighbor. Therefore, W is not a vertex cut of size $k-1$. This implies that G is k-connected. ■

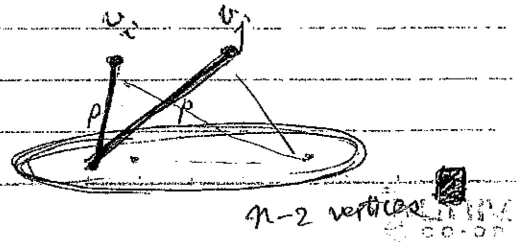
Not only the graph is with high connectivity, the graph does have very small diameter.

Theorem 82 Almost all graphs are of diameter 2.

Proof. Let $X_{i,j}$ be the indicator random variable such that

$$X_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ do not have a common neighbor; and} \\ 0 & \text{otherwise.} \end{cases}$$

So, the random variable X of "no two vertices have a common neighbor" is equal to $\sum_{i\neq j} X_{i,j}$. Hence, $E(X) = \sum_{i\neq j} E(X_{i,j}) = \binom{n}{2}\cdot(1-p^2)^{n-2} \to 0$. This implies that $E(X) \to 0$ and we conclude that almost every pair of distinct vertices have a common neighbor. ■



(*) We may replace p by $p(n)$. Then, we have a more complicat. situation to consider the random graphs.

(**) We may also consider the probability of edges based on the vertices they are incident to. That is, the next edge will come from somewhere near a vertex with larger degree.

Problem Show that almost all graphs there is a unique vertex with maximum degree. (Ref. JCT(B) 23, 255-257 (1977), P. Erdős and R. Wilson).

(***) If this is true, then almost all graphs are of Class 1, since Vizing did prove that a Class 2 graph contains at least three major vertices (degree $\Delta(G)$).

For convenience in referring this result, we put it as a theorem.

Theorem 83 For $p \in (0,1)$, almost all graphs obtained in Model A is of Class 1.

Proof.

Step 1. Prove that almost all graphs have a unique major vertex. (Exercise)

Step 2. By Vizing's result, every Class 2 graph has at least three major vertices, we conclude the proof.

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Note

On the Chromatic Index of Almost All Graphs

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Vizing has shown that if G is a simple graph with maximum vertex-degree ρ, then the chromatic index of G is either ρ or $\rho + 1$. In this note we prove that almost all graphs have a unique vertex of maximum degree, and we deduce that almost all graphs have chromatic index equal to their maximum degree. This settles a conjecture of the second author (in "Proceedings of the Fifth British Combinatorial Conference 1975").

Let G be a simple graph (that is, a graph without loops or multiple edges), and suppose that the maximum vertex-degree of G is ρ. Vizing [8] has shown that the chromatic index $\chi'(G)$ of G must be equal either to ρ (in which case we say that G is of *class one*) or to $\rho + 1$ (in which case G is of *class two*).

Limited numerical evidence seems to suggest that most graphs are of class one; for example, it is shown in [1] that of the 143 connected graphs with not more than six vertices, only eight are of class two. The object of this note is to prove this result in general.

THEOREM. *Almost all graphs are of class one.*

(In this note, to say that "almost all graphs have a given property" means that if $P(n)$ is the probability that a random graph with n vertices has that property, then $P(n) \to 1$ as $n \to \infty$; in other words, if U_n is the number of graphs with n vertices having that property, and if V_n is the total number of graphs with n vertices, then $U_n/V_n \to 1$ as $n \to \infty$. Further results on random graphs may be found in [2, 3].)

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In order to establish this theorem, we shall need the following lemma, which is of considerable interest in its own right:

LEMMA. *Almost all graphs have a unique vertex of maximum degree.*

Proof of lemma. We first prove the corresponding result for labeled graphs.

If G is a random labeled graph with n vertices, then the probability that a given vertex has degree k is $\binom{n-1}{k} 2^{1-n}$, since each edge of G appears with probability $\frac{1}{2}$.

If $k = \frac{1}{2}(n-1) + t$ (say), then by a standard asymptotic argument for the binomial distribution (see, for example, [4, pp. 179–180]), we have

$$\binom{n-1}{k} 2^{1-n} = (1 + o(1))(\tfrac{1}{2}\pi n)^{-1/2} e^{-2t^2/n}. \tag{1}$$

It follows from the Inclusion–Exclusion Principle (as used, for example, in [6, pp. 71–72]) that for almost all graphs G, the maximum vertex-degree of G is equal to

$$\tfrac{1}{2}(n-1) + \tfrac{1}{2}(n \log n)^{1/2} + o(n \log n)^{1/2}$$

and hence that G almost surely has a vertex of degree at least

$$\tfrac{1}{2}(n-1) + \tfrac{1}{2}\{(1-\epsilon)\, n \log n\}^{1/2}, \tag{2}$$

for any given $\epsilon > 0$.

To prove the lemma for labeled graphs, it now suffices to prove that if

$$k > \tfrac{1}{2}(n-1) + \tfrac{1}{2}\{(1-\epsilon)\, n \log n\}^{1/2}, \tag{3}$$

then G almost surely does not have two vertices both of degree k.

But the probability that two given vertices both have degree k is $(1 + o(1))\binom{n-1}{k}^2 2^{2-2n}$, and so it is enough to prove that

$$n^2 \sum{}' \binom{n-1}{k}^2 2^{2-2n} \to 0, \qquad \text{as } n \to \infty, \tag{4}$$

where the prime indicates that the summation extends only over those values of k satisfying (3). But (4) follows from (1) by a simple calculation, and so the lemma is proved for labeled graphs.

To deduce the corresponding result for unlabeled graphs is now a simple matter. Since almost all unlabeled graphs with n vertices can be labeled in $n!$ ways, and since (by a result of Pólya) the number of unlabeled graphs with n

vertices is asymptotically equal to $(n!)^{-1}$ times the number of labeled graphs with n vertices (that is, $2^{n(n-1)/2}$), it follows that every property which is true for almost all labeled graphs is simultaneously true for almost all unlabeled graphs, and conversely. (This is the "Metatheorem" in [5, Chapter 9], to which the reader is referred for a further discussion of this type of argument.) The result for unlabeled graphs therefore follows from the result for labeled graphs, thereby completing the proof of the lemma.

To deduce the theorem from the lemma, it is sufficient to prove that if a graph G has only one vertex of maximum degree, then G is necessarily of class one. But this follows immediately from a result of Vizing [9] which states that every graph of class two has at least three vertices of maximum degree. This completes the proof of the theorem.

We conclude this note with the following corollary:

COROLLARY. (i) *Almost all connected graphs are of class one.*

(ii) *Almost all 2-connected graphs are of class one.*

(iii) *Almost all Hamiltonian graphs are of class one.*

Proof. (i) follows since almost all graphs are connected [5, p. 206].

(ii) follows since almost all graphs are 2-connected [5, p. 207].

(iii) follows since almost all graphs are Hamiltonian [7].

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