

(*) There exists a procedure to construct all the graphs G with $\chi(G) \geq k$.

(**) \mathcal{G}_k is a collection of k -constructible graphs if $G \in \mathcal{G}_k$ can be constructed recursively by the following steps.

(i) K_k is k -constructible.

(ii) If G is k -constructible and $x, y \in V(G)$ are non-adjacent, then $(G+xy)/xy$ is k -constructible.

(iii) If G_1 and G_2 are k -constructible such that $V(G_1) \cap V(G_2) = \{x\}$, $xy_1 \in E(G_1)$, $xy_2 \in E(G_2)$, then $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k -constructible, see Figure 41.

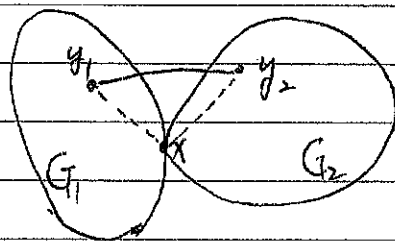


Figure 41. Hajós construction.

(*) If G is k -constructible, then $\chi(G) \geq k$.

(i) is trivial (ii) If $(G+xy)/xy$ uses less than k colors, then by coloring x and y with the same color, we have $\chi(G) < k$.

(iii) If the graph obtained uses less than k colors, then either

$\chi(G_1) < k$ or $\chi(G_2) < k$ depending on whether $\varphi(y_1) \neq \varphi(x)$ or

$\varphi(y_2) \neq \varphi(x)$. Since y_1 and y_2 receive distinct colors, one of the

above two conditions must hold.

Theorem 66. (Hajós, 1961)

Let G be a graph. Then, $\chi(G) \geq k$ if and only if G has a k -constructible subgraph.

Proof. (\Leftarrow) It has been explained above.

(\Rightarrow) Suppose not; then $k \geq 3$. Let G be a maximal counterexample,

i.e. G is of maximum size such that G does not contain a k -constructible

subgraph. Now, G can not be a complete r -partite graph.

For otherwise, $\chi(G) \geq k$ implies that $r \geq k$ and then G contains

a k -constructible K_k . (Contract each partite set.) Hence there

exist vertices x, y_1 and y_2 such that $xy_1 \notin E(G)$, $xy_2 \notin E(G)$ but

$y_1y_2 \in E(G)$. By assumption of the maximality of G , both $G+xy_1$

and $G+xy_2$ contain k -constructible subgraphs, say H_1 and H_2 ;

moreover, $xy_1 \in E(H_1)$ and $xy_2 \in E(H_2)$.

Let $H_2 - H_1$ denote the graph $\langle V(H_2) \setminus V(H_1), E(H_2) \setminus E(H_1) \rangle$ and H'_2 is an isomorphic copy of H_2 such that $V(H'_2) \cap V(G) = \{x\} \cup (V(H_2) \setminus V(H_1))$,

see Figure 42. So, $V(H_1) \cap V(H'_2) = \{x\}$. Now, since $H'_2 \cong H_2$, let

$\varphi: H_2 \rightarrow H'_2$ be an isomorphism. By (iii) $H_1 \cup H'_2 = xy_1 - x - \varphi(y_2) + y_1 - \varphi(y_2)$

is k -constructive, let this graph be H . Now, for each vertex v'

in $V(H'_2) \setminus V(G)$, there exists a v such that $v' = \varphi(v)$. Furthermore

vv' is not an edge of H . By (ii), we can identify v and v' and

obtain a k -constructive subgraph of G after identifying all

vertices in $V(H_2) \setminus V(H_1)$. ■

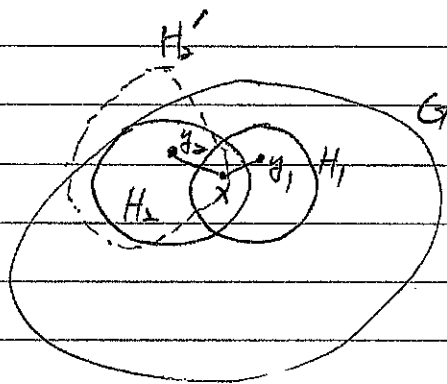


Figure 42

(*) A k -edge-coloring is a mapping $\pi: E(G) \rightarrow \{1, 2, \dots, k\}$ such that incident edges receive distinct images (colors).

(*) $\chi'(G) = \min\{k \mid G \text{ has a } k\text{-edge-coloring}\}$. (Chromatic index of G .) If $\chi'(G) = k$, then G is h -edge-colorable for each $h \geq k$.

Theorem 67. (Vizing, 1964)

If G is a simple graph, then $\Delta(G) \leq \chi' \leq \Delta(G) + 1$.

Proof. The left hand inequality is easy to see, we prove the right hand inequality. By induction on $\|G\|$. We shall prove

that G has a $(\Delta(G) + 1)$ -edge-coloring for G and the assertion (coloring in short)

is true for smaller sizes, i.e., for each $e \in E(G)$, $G - e$ has

a coloring. $\Downarrow \pi$ (Let $e = xy$.)

First, we observe that since each vertex v is of degree at most

$\Delta(G)$, a color is missing around v . Second, if α and β be

two colors used in the coloring, then α and β induce a ^{sub-}graph _(a)

with components either paths or even cycles.

Finally, if G has no colorings using $\Delta(G)+1$ colors, then for each edge xy and any coloring of $G-xy$, there exists an β - α path from y ends in x provided α is missing at x and β is missing at y . See Figure 4.3 for missing colors.

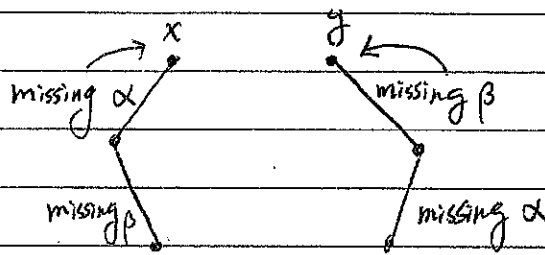


Figure 4.3

(*) If α - β path does not connect x and y , then we may recolor one of the path (α, β) , to obtain a coloring of G using $\Delta(G)+1$ colors.

Clearly, if x and y are missing the same color, then we can use that color to color xy and obtain a $\Delta(G)+1$ coloring of G .
(By induction)

(*) Claim: There is a way to recolor some edges in $G-xy$ such that x and y miss the same color.

Outline of proof

Let $M(y)$ denote the colors missing at y , and $c_1 \in M(y)$.

Now, consider $M(x)$. If $c_1 \in M(x)$, then color xy by c_1 , results

in a $\Delta(G)+1$ coloring of G . (The claim holds.)

Hence, $c_1 \notin M(x)$, let $c_0 \in M(x)$ and $\pi(xy_1) = c_1$. (See Figure 44)

Then, consider $M(y_1)$ and let $c_2 \in M(y_1)$. Note that $c_2 \notin M(x)$.

If $c_2 \in M(x)$, then we let $\pi(xy_1) = c_2$. Thus, c_1 becomes a missing

color in $M(x)$, the coloring for xy is available, $\pi(xy) = c_1$. This

fact will continue: $c_1 \notin M(x) \Rightarrow \exists y_2$, s.t. $\pi(xy_2) = c_2$; and

then $c_3 \in M(y_2)$, $\pi(xy_3) = c_3, \dots, c_{i+1} \in M(y_i)$, $\pi(xy_{i+1}) = c_{i+1}$.

Since we only have $\Delta(G)+1$ colors, there exists an l such that

$\pi(xy_{l+1}) = c_{l+1} \in \{c_1, c_2, \dots, c_l\}$. W.L.O.G., let $c_{l+1} = c_k, k \in \{1, 2, \dots, l\}$.

Now, we have several cases to consider depending on whether

$c_0 \in M(y_l)$ or $c_0 \notin M(y_l)$.

(*) If $c_0 \in M(y_l)$,
then color xy_l
by c_0 .
This implies
that $c_1 \in M(x)$.

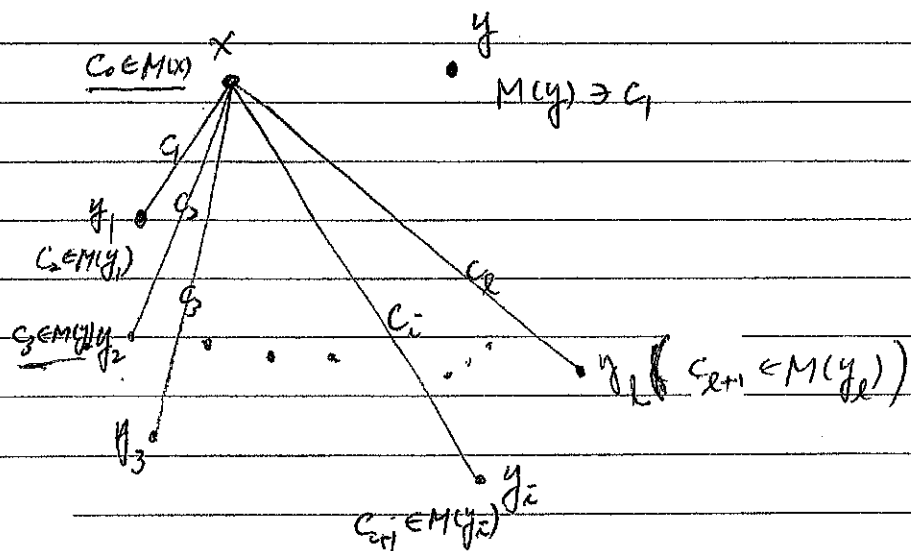


Figure 44 (Fan sequence!)

(a) $c_0 \in M(y_k)$.

Then, let $\pi(xy_k) = c_0$, $\pi(xy_{k-1}) = c_1$, \dots , $\pi(xy_1) = c_2$. This implies that $c_1 \in M(x)$ and the proof follows.

(b) $c_0 \notin M(y_k)$.

Since $c_{k+1} = c_k$, $c_k \in M(y_k)$. Now, consider $c_k - c_0$ path starting from y_k .

(i) It is a $y_k - y_k$ path. Since $\pi(xy_k) = c_k$, we may recolor them to a $c_0 - c_k$ path starting from y_k . (Note here that c_0 occurs in an edge incident to y_k . By the fact that the last color is c_k , both c_0 and c_k occur an even number of times.) Now, since $\pi(xy_k) = c_0$, the recoloring of $xy_1, xy_2, \dots, xy_{k-1}$ gives $c_1 \in M(x)$, we have the proof.

(ii) It is a $y_k - y_{k-1}$ path. Since $c_k \in M(y_{k-1})$, this path is ended with color c_0 . That is to say c_0 is also available for xy_{k-1} (not only c_{k-1}). Hence, we color xy_{k-1} with c_0 instead of c_{k-1} , the proof follows by a similar recoloring process.

(iii) It is a $y_{i-1} - y_i$ path, $i \notin \{k-1, k\}$.

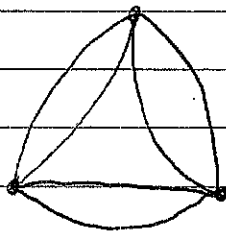
Then, either c_i or c_{i+1} will be available for $x y_i$ and the proof follows by recoloring process. ■

(*) Based on the same proof technique, we also have a stronger result of Vizing's Theorem.

Theorem 67' (Vizing, 1964)

If G is a multigraph with multiplicity η , then $\chi'(G) \leq \Delta(G) + \eta$.

(*) The following graph has $\Delta(G) = 4$ and $\eta = 2$.



Definition (Class 1 and Class 2)

A graph (simple) is of Class 1 if $\chi'(G) = \Delta(G)$ and of Class

2 if $\chi'(G) = \Delta(G) + 1$.

(König, 1916)

✓ Theorem 68, A bipartite graph is of Class 1.

Proof. (1st)

By induction on $\|G\|$. Let $xy \in E(G)$ and $G - xy$ can be edge-colored with $\Delta(G)$ colors. Now, since $\deg_{G-xy}(x) < \Delta(G)$ and $\deg_{G-xy}(y) < \Delta(G)$,

a color is missing at x and also a color is missing at y . Let them

be α and β respectively. Clearly, $\alpha \neq \beta$, and β occurs around x and

α occurs around y . Now, we adapt the idea in proving Vizing's

Theorem, let P be a ^{longest} α - β path from x : $\overset{\beta}{\bullet} - \overset{\alpha}{\bullet} - \overset{\beta}{\bullet} - \dots$.

First, if P is an x - y path and the last edge has color α , then

P is a path of even length. Hence, $P \cup \{xy\}$ is an odd cycle. $\rightarrow \leftarrow$
 G is bipartite.

Hence, x and y are in different components induced by the set of

edges colored α and β . Now, we recolor all the edges of P by

interchanging α and β . This gives a coloring in which β is missing

at x and also at y . By coloring xy with β , we obtain a Δ -edge-

coloring of G . ■

2nd proof

Lemma Let G be a bipartite graph. Then, there exists a ^{$\Delta(G)$ -}regular

bipartite graph $\tilde{G} \geq G$. (Exercise)

By Lemma \tilde{G} is a $\Delta(G)$ -regular bipartite graph and thus \tilde{G}

can be decomposed into $\Delta(G)$ perfect matchings by König's Theorem.

This implies that $\chi'(\tilde{G}) = \Delta(G)$. Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) \leq \Delta(G)$.

Hence, we conclude the proof.

(*) A graph G is said to be overfull if $\|G\| > \lfloor \frac{|G|}{2} \rfloor \cdot \Delta(G)$.

(**) If G is overfull, then G is of Class 2.

(*) If G is overfull, then $|G|$ is odd.

Theorem 69

Petersen graph is of Class 2.

Proof. If G is the Petersen graph and $\chi'(G) = 3$, then G can be decomposed into 3 1-factors: F_1, F_2 and F_3 (3 color classes).

Now, consider the set of 5 link-edges e_1, e_2, e_3, e_4 and e_5 , see

Figure 45.

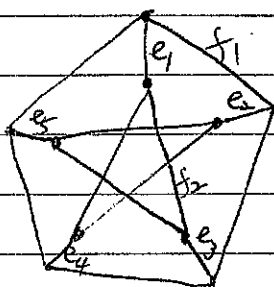


Figure 45 Petersen graph

At least one of F_1, F_2 and F_3 will contain at least two link-edges

by Pigeon-hole principle, let it be F_1 . Clearly, F_1 can not contain all the 5 link-edges. For otherwise, two C_5 's is the union of F_1 and

F_2 which is impossible. So, there are three cases to consider.

$$(i) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 4$$

Let e_1 be the edge not in F_1 . But, now all the edges ^{of $G - e_1$} not in

$\{e_2, e_3, e_4, e_5\}$ are incident to an edge of $\{e_2, e_3, e_4, e_5\}$. So, no other

edge can be chosen for F_1 .

$$(ii) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 3$$

Let e_1 and e_2 be the edges not in F_1 . Then, other than link-edges,

we can choose at most one more edge f_1 . The case e_1 and e_2 ^{are} not

in F_1 has similar conclusion (only f_2 is available).

$$(iii) |F_1 \cap \{e_1, e_2, \dots, e_5\}| = 2$$

This case comes out that we can find two more edges which

not link-edges. ■

(*) The proof of Theorem 69 implies that the Petersen graph contains no Hamilton cycles.

Proof. If G contains a Hamilton cycle C , then $\chi'(G) = 3$ by coloring the cycle with two colors and $G - C$ (1-factor) with another color. ■

Theorem 70

A 3-regular planar graph G is of Class 1.

Proof. Let G be embedded in S_0 . Then, by 4-color Theorem,

G is 4-face-colorable (or 4-map-colorable). Let the 4 colors

used be obtained from the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$. Since each edge

is in the boundary of two adjacent faces, let the edge be colored

by $(a_1, b_1) \oplus (a_2, b_2)$ where (a_1, b_1) and (a_2, b_2) are the colors of these

two adjacent faces. As a conclusion, we obtain a 3-edge-coloring

of G , since $(0, 0)$ will not be used. The coloring is proper since

three adjacent faces will receive three different colors, see

Figure 4b. ■

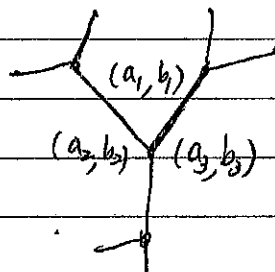


Figure 4b.

Theorem 21 (Equitable edge-coloring)

If G has a k -edge-coloring, then G has an equitable edge-coloring, i.e., for any two $i, j \in \{1, 2, \dots, k\}$, $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$.

Proof. If there exist i and j such that $|f^{-1}(i)| - |f^{-1}(j)| \geq 2$, then we consider the graph H induced by the set of edges colored i and j .

Then, H is a subgraph of G such that each component of H is either a path or an even cycle. Since i occurs more times than j , there exists an i - j path: $\overset{i}{\bullet} - \overset{j}{\bullet} - \overset{i}{\bullet} - \overset{j}{\bullet} - \dots - \overset{i}{\bullet}$ whose

end edges are colored i . Now, by switching the colors on this path,

we obtain a new edge coloring of G such that i occurs one less

time and j occurs one more. It turns out that we can obtain

an k -edge-coloring s.t. $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$. As a consequence,

we are able to adjust all of them and obtain an equitable

k -edge-coloring. ■

(*) This theorem is also not difficult to prove, but very useful.

(o) Without using 4CT, the proof of Theorem 70 is very difficult.

(*) It was conjecture that if G is planar and $\Delta(G)$ is large enough, say,

Theorem 22

The complete graph K_n is of Class 2 if and only if K_n is overfull or equivalently n is odd.

Proof. First, we claim that for each $m \geq 1$, K_{2m} is of Class 1. It suffices to give a $(2m-1)$ -edge-coloring of K_{2m} . For convenience, let

$V(K_{2m}) = \mathbb{Z}_{2m} = \{0, 1, 2, \dots, 2m-1\}$. For each color $i \in \{1, 2, \dots, 2m-1\}$,

let the set of edges colored i be

$$F_i = \{ (0, i), (i+1, i-1), (i+2, i-2), \dots, (i+m-1, i-m+1) \} \pmod{2m-1}.$$

See Figure 46 for an example of $m=5$ and $i=3$.

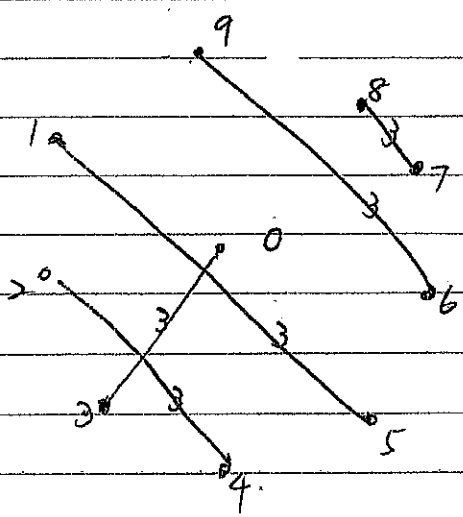


Figure 46. $\chi(K_{10})=9$.

Since $\Delta(K_{2m}) = 2m-1$, $\chi'(K_{2m}) = 2m-1$.

Now, by deleting 0 in K_{2m} , we obtain a $(2m-1)$ -edge-coloring of K_{2m-1} . On the other hand, it is not difficult to check that K_{2m-1} is overfull for $m \geq 2$, this concludes that $\chi'(K_{2m-1}) > \Delta(K_{2m-1}) = 2m-2$. ■

(*) This theorem is not difficult to prove, but it is very useful in the construction of "Combinatorial Designs".

(**) Equivalently, K_{2m} can be decomposed into $2m-1$ 1-factors, which is also known as a 1-factorization of K_{2m} .

(***) If G is an r -regular graph and $\chi'(G) = r$, then G has a 1-factorization.

(****) It was conjectured that if G is r -regular and $r \geq \frac{|G|}{2}$, then G has a 1-factorization or equivalently $\chi'(G) = r$.

Theorem 21' (D. Hoffman et al.)

A complete multipartite graph G is of Class 2 if and only if G is overfull.

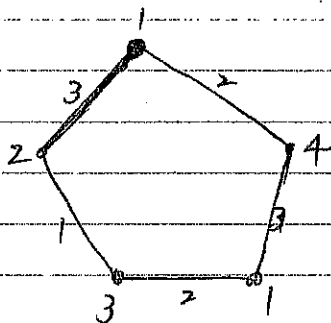
(*) Total coloring

A k -total coloring of a graph G is a mapping

$$\psi: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\} \text{ such that}$$

- (i) adjacent vertices receive distinct images,
- (ii) incident edges receive distinct images, and
- (iii) each vertex has a distinct image with \nearrow the images of its incident edges.

e.g.



A 4-total coloring of C_5 .

(*) $\chi''(G) = \min \{ k \mid G \text{ has a } k\text{-total coloring} \}$.

(Total chromatic number)

Theorem 23

$$\chi''(K_{2n+1}) = \chi''(K_{2n}) = 2n+1.$$

$$\chi''(K_5) = 5 \text{ (?)}$$

Proof. $\chi''(K_{2n+1})$ can be obtained by using $\chi'(K_{2n+1}) = 2n+1$.

TCC Conjecture $\chi''(G) \leq \Delta(G) + 2$.

12-17

Note that $\chi''(G) \geq \Delta(G) + 1$. As to the total coloring of K_{2n} , we claim that $2n$ colors are not enough.

Observe that each color class has at most one vertex and $n-1$ edges. So, $2n$ color classes will contain at most $2n$ vertices and $2n(n-1)$ edges. Hence, ^{there are} $2n^2$ elements (vertices and edges) in total. But, K_{2n} has $2n + \frac{2n(2n-1)}{2}$ elements to color, which is $2n^2 + n$. Clearly, $2n$ color is not enough. Since K_{2n+1} is $(2n+1)$ -total colorable, K_{2n} is also $(2n+1)$ -total colorable. The proof follows. \blacksquare

(c) Based on TCC Conjecture, a graph G is called Type 1 if $\chi''(G) = \Delta(G) + 1$ and Type 2 otherwise.

Theorem 24. $K_{m,n}$ is of Type 1 if and only if $m \neq n$.

Proof. (\Rightarrow) If $m = n$, then there are $2n + n^2$ elements to color. Since each color class contains at most n elements, $\Delta(G) + 1 = n + 1$ colors are not enough, $n(n+1) < 2n + n^2$. Hence, $\chi''(K_{n,n}) \geq n + 2$, and thus $K_{n,n}$ is ^{not} of Type 1.

(\Leftarrow)

On the other direction, let $m = n+k$. Now, $\Delta(K_{m,n}) = n+k$.

By the edge-coloring of $K_{m,n}$, we have an $n \times (n+k)$ Latin rectangle based on $\{1, 2, \dots, n+k\}$, see Figure 47. Since $k \geq 1$, we may extend this rectangle to $(n+1) \times (n+k)$ and the last row can be used to color the vertices of

A. Finally, color all vertices of B by one extra color, we have

$$\chi^v(K_{m,n}) \leq n+k+1 = \Delta(K_{m,n})+1. \quad \blacksquare$$

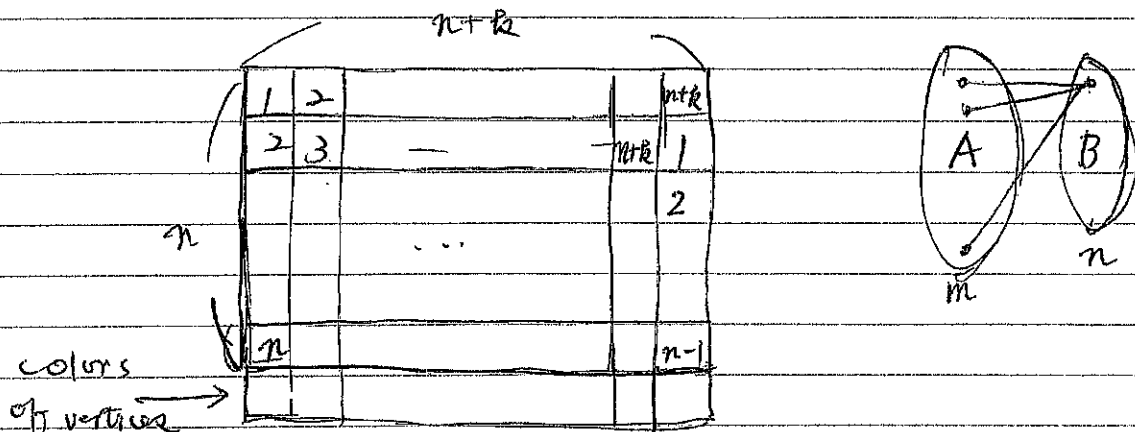
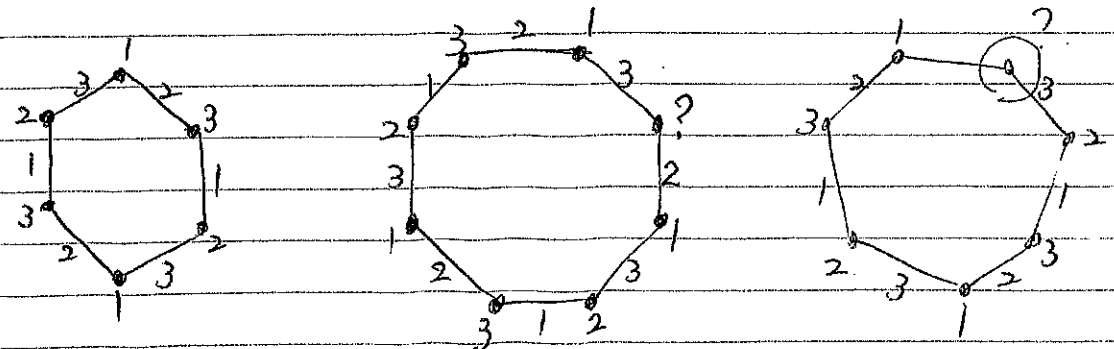


Figure 47, An edge-coloring of $K_{m,n}$.

(*) A cycle C_n is of Type 1 if and only if $n \equiv 0 \pmod{6}$.



If we use three colors, then starting from one vertex and one edge, all the colors of the others are forced!

(*) The deficiency of a graph G , $\text{def}(G)$, is defined as $\sum_{v \in V(G)} (\Delta(G) - \text{deg}(v))$.

(**) G is conformable if G has a vertex coloring $\varphi: V(G) \rightarrow \{1, 2, \dots, \frac{\Delta(G)}{2} + 1\}$

such that $\text{def}(G) \geq |\{i \mid |G| - |\varphi^{-1}(i)| \equiv 1 \pmod{2}\}|$.

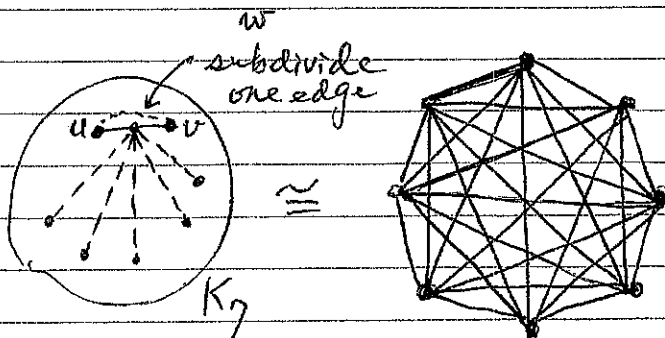
Theorem 75

Let S_i be a star with i edges. Then, $K_{2n} - S_1 - S_{2n-3}$ is of Type 2.
 $\parallel G(1, 2n-1)$

Proof. Assume that $G(1, 2n-1)$ is of Type 1, i.e., there exists a total coloring ψ of $G(1, 2n-1)$ using $\Delta(G)+1 = 2n-1$ colors. Let uv be the edge subdivided, see Figure 48. First, if $\psi(u) = \psi(v)$, then let $\psi(w) = 1$, $\psi(uw) = 2$ and $\psi(wv) = 3$. Let r_j the number of vertices in which j occurs in either v or an edge incident to v . Hence, $\sum_{j=0}^{2n-2} r_j = |K_{2n}| + 2 \parallel G(1, 2n-1) \parallel$.

(?) Now, 0 occurs in at most $2n-2$ vertices. Hence, $\sum_{j=0}^{2n-2} r_j \leq (2n-2) + 3 \cdot (2n) + (2n-5) \cdot (2n-1) = 4n^2 - 4n + 3 < |K_{2n}| + 2 \parallel G(1, 2n-1) \parallel$.

$\rightarrow \epsilon$



$n=4$

$\leftarrow G(1, 5)$

Figure 48. $G(1, 5)$.

On the other hand, if $\psi(u) \neq \psi(v)$, a similar argument shows that $2n-1$ colors are not enough. Hence, $G(1, 2n-3)$ is of Type 2. ■

(*) If G is of Type 1, then G is conformable.

Proof. If there exists a color i such that $|G| - |\psi_{V(G)}^{-1}(i)| \equiv 1 \pmod{2}$, then i occurs in at most $|G|-1$ vertices. Since every color occurs around a major vertex, $\Delta(G)+1$ colors are not enough if $\text{def}(G) < |\{i \mid |G| - |\psi_{V(G)}^{-1}(i)| \equiv 1 \pmod{2}\}|$. So, if G is not conformable, then G is not ^{of} Type 1. ■

Conjecture (Chetwynd and Hilton, 1988)

Let G be a simple graph with $\Delta(G) \geq \lfloor \frac{|G|+1}{2} \rfloor$. Then, G is of Type 2 if and only if there exists a non-conformable subgraph H of G such that $\Delta(H) = \Delta(G)$ (and G is not the Chen and Fu graph). ↑ 12.7.10.1

(000) The ^{original} conjecture was disproved by using Theorem 7.5.

(*) The graph $G(1, 2n-3)$ is known as Chen and Fu graph.