Graph Theory Lecture 11 Nov. 23, 24

Vertex-Coloring

(∘) k-coloring (proper) : $\varphi : V(G) \to \{1,2,3,\cdots,k\}$ s.t. $uv \in E(G) \Rightarrow \varphi(u) \neq \varphi(v)$.

(∘) $\chi(G) = \min\{k \mid G \text{ has a } k\text{-coloring}\}$ (Chromatic number of G)

(∘) G is n-critical (chromatically) if $\chi(G-v) < \chi(G)$ for each $v \in V(G)$.

(∘∘) Every graph G has an n-critical induced subgraph H.

(∘∘∘) $\chi(G) \geq \omega(G)$ (Clique number), $\alpha(G) \geq \left\lceil \frac{|G|}{\chi(G)} \right\rceil$.
(Independence number)

Theorem 56

Every critically n-chromatic graph, $n \geq 2$, is $(n-1)$-edge-connected.
(n-critical) (G) $\Downarrow$ $\delta(G) \geq n-1$.

Proof. First, if $n=2$, then $G \cong K_2$ and thus G is 1-edge-connected.

If $n=3$, then $G \cong C_{2m+1}$, $m \geq 1$, (?) and G is 2-edge-connected.

Let $n \geq 4$ and assume that G is not $(n-1)$-edge-connected.

Hence, $V(G) = V_1 \cup V_2$ such that $|\langle V_1, V_2 \rangle| < n-1$. Let $G_1 = \langle V_1 \rangle_G$
$(\leq n-2)$

and $G_2 = \langle V_2 \rangle_G$. Now, both of them are $n-1$ colorable since

(∘) If $n=2$, then G is a bipartite graph.
($n=3$) (contains an odd cycle)

Now, consider the vertices incident to the edges in $\langle V_1, V_2 \rangle$. If for each edge $uv \in \langle V_1, V_2 \rangle$, $\varphi_1(u) \neq \varphi_2(v)$, then G has an $(n-1)$-coloring, a contradiction. Thus, assume that for some edges $uv \in \langle V_1, V_2 \rangle$, $\varphi_1(u) = \varphi_2(v)$. (We shall $\underbrace{\text{permute the colors of } G_1}_{\varphi_1'}$ in order that for each $uv \in \langle V_1, V_2 \rangle$, $\varphi_1'(u) \neq \varphi_2(v)$.

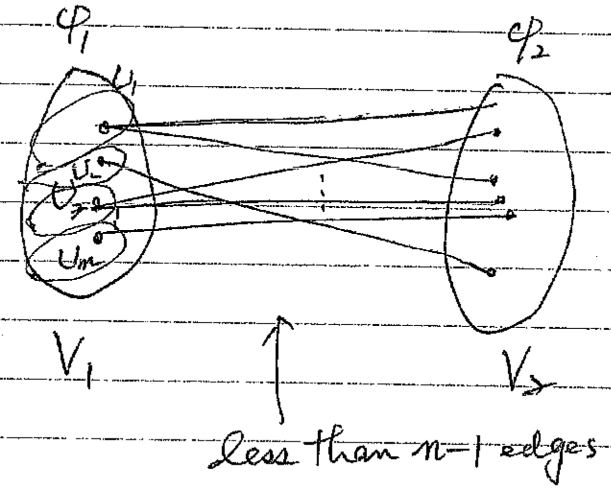
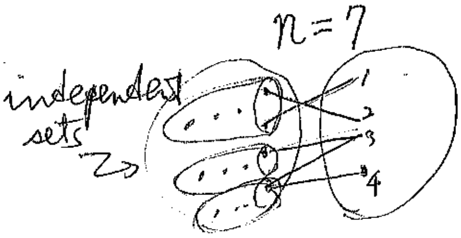


Figure 37.

Let $U_1, U_2, \cdots, U_m$ be the subsets of V_1 such that $\varphi_1^{-1}(i) = U_i$, $i = 1, 2, \cdots, m \ (\leq n-2)$ and there is at least one edge joining U_i and $V(G_2)$ for each i. Furthermore, let n_i be the number of vertices in U_i which are incident to a vertices of $V(G_2)$. Hence, $\sum_{i=1}^{m} n_i \leq n-2$.



Now, we start a process to recolor the vertices in V_1. Starting U_1. If $\forall\ x \in U_1$, $\varphi_1(x)$ has distinct colors with the colors of those vertices in V_2 which are incident to U_1, then we go to consider U_2. Otherwise, $\varphi_1(x) = \varphi_2(y)$ for some $x \in U_1$, and $xy \in \langle V_1, V_2 \rangle$. In this case, we permute the colors of $U_1, U_2, \cdots, U_m$ (It was $1, 2, \cdots, m$.) such that the color used for the vertices in U_1 is distinct from the colors of vertices in V_2 which are incident to U_1, there are n_1 of them. Since $\sum_{i=1}^{m} n_i \le n-2$, $n-1-n_1 > 0$ and thus there exists a color for U_1.

Following this idea, we consider U_2. If there are vertices x in U_2 such that $\varphi_1(x) = \varphi_2(y)$ for some $xy \in \langle V_1, V_2 \rangle$, then permute the colors used in $U_2, U_3, \cdots, U_m$ where the color for U_1 is fixed. Again, since $n-2-n_2 \ge (n-1)-n_1-n_2 > 0$, a color for U_2 is available. Continuing this process, we end it up with an $(n-1)$-coloring of G, a contradiction to $\chi(G) = n$. ∎

(c) If G is n-critical, then $\delta(G) \ge n-1$.

Theorem 52.

Let $k = \max_{H \preceq G} \delta(H)$. Then, $\chi(G) \le k+1$. ($H \preceq G$, induced subgraph)

Proof (1st). Let $\chi(G) = n$ and H' be an n-critical induced subgraph of G. Then, $\delta(H') \ge n-1$. Since $\max_{H \preceq G} \delta(H) \ge \delta(H') \ge n-1$, $k \ge n-1$ and thus $\chi(G) = n \le k+1$. ∎

2nd proof. (G is a (p,q)-graph.)

Since $k = \max_{H \preceq G} \delta(H)$, $\delta(G) \le k$. Let $x_p \in V(G)$ and $\deg_G(x_p) \underset{\le k}{=} \delta(G)$. Moreover, let $G_{p-1} = G - x_p$. Again, G_{p-1} has a vertex of degree at most k. So, we obtain a sequence of induced subgraphs, $G = G_p \succeq G_{p-1} \succeq \cdots \succeq G_1$ such that $\delta(G_i) \le k$ for $i = p, p-1, \cdots, 1$ such that $x_i \in V(G_i)$. Hence, we obtain a sequence $\langle x_1, x_2, \cdots, x_p \rangle$ such that x_{j+1} is incident to at most k vertices in $\langle \{x_1, x_2, \cdots x_j\} \rangle_G$. This implies that we can use greedy algorithm to color G starting from x_1, and then $x_2, \cdots, x_p$. All we need is at most $k+1$ colors. Hence, $\chi(G) \le k+1$. ∎

Theorem 58 (Brooks, 1941)

Let G be a connected graph which is neither a complete graph nor an odd cycle. Then, $\chi(G) \le \Delta(G)$.

Proof. By induction on $|G| = p$. We may assume that the graph we consider is 2-connected and Δ-regular where $\Delta \ge 3$. (?) (Note that a 2-regular connected graph G with $\chi(G) = 3$ is an odd cycle.) (*)假如有一點 degree $< \Delta(G)$，去掉該點再用歸納假設即可。)

First, if G is 3-connected, let x_p be any vertex such that $\langle N_G(x_p) \rangle_G$ is not a complete subgraph of G. Such an x_p does exist since G is not a complete graph. Let x_1 and x_2 be two vertices in $N_G(x_p)$ such that $x_1 \nsim_G x_2$. Now, we may construct a sequence corresponding to $V(G)$. Choose $x_{p-1} \in N_G(x_p) \setminus \{x_1, x_2\}$ (3-connected). Then, x_{p-2} is adjacent to either x_p or x_{p-1}. As a consequence, we have a sequence $\langle x_1, x_2, \cdots, x_p \rangle$ such that x_i is incident to at least one vertex in $\{x_{i+1}, x_{i+2}, \cdots, x_p\}$. Now, we use the greedy algorithm to obtain the Δ-coloring.

<u>Greedy</u> <u>Coloring</u> (Vertex) 11-5'

$$|G| = p$$

<u>Step 1</u> Determine an order of vertices which are to be colored.

$\langle x_1, x_2, \cdots, x_p \rangle$

For $i < p$, x_i is incident to a vertex in $\{x_{i+1}, x_{i+2}, \cdots, x_p\}$.

(註) 連通圖的點集合可以 "Ordered". (Theorem 11)

<u>Step 2</u> Starting from x_1, choose a color which is available to color the vertices. (If we have $\Delta(G)$ colors, then we can color all vertices except possibly the last vertex.)

(*) 因為每个點的鄰居一直維持有至少一个點尚未上色。

<u>Fact 1.</u> If $\deg(x_p) < \Delta(G)$, then $\Delta(G)$ colors are enough.

<u>Fact 2.</u> If G is $\Delta(G)$-regular, then at most $\Delta(G)+1$ colors are enough.

<u>Fact 3.</u> $\Delta(G)+1$ colors are needed if G is an odd cycle or a complete graph.

<u>Fact 4.</u> If we can precolor two non-<u>adjacent</u> vertices of x_p by using the same color, then $\Delta(G)$ colors are enough.

(*) This is the <u>main idea</u> of proving Brooks' Theorem.

(*) If G is properly colored by using k colors, then the vertex set $V(G)$ can be partitioned into k independent vertex sets $V_1, V_2, \cdots, V_k$. (同色之间没有边)

Definition

A graph G is said to be $(i_1, i_2, \cdots, i_k)$-colorable if the maximum degree of the induced subgraph $\langle V_j \rangle_G$ is i_j, where i_j is a non-negative integer for $j = 1, 2, \cdots, k$.

(∘) If G is k-colorable, then G is $(\underbrace{0, 0, \cdots, 0}_{k \text{ tuples}})$-colorable.

(∘) A bipartite graph is 2-colorable and thus $(0,0)$-colorable.

(**) If $V(G)$ of G can be partitioned into two subsets V_1 and V_2 such that $\langle V_1 \rangle_G$ and $\langle V_2 \rangle_G$ are bipartite graphs, then G is 4-colorable.

(Fact) If G is $(1,0,0)$-colorable, then G is 4-colorable.

Chromatic Theory : Starting from the proof of 4-color theorem. (It was proved first by Appel and Haken at 1976 by the aid of "computers". A written proof is still missing.)

Second, let G be 2-connected (but not 3-connected), i.e. $\kappa(G)=2$. Let S be a cut set with two vertices and $x_p \in S$. Hence, $G - x_p$ has a cut vertex, see Figure 38. Let x_1 and x_2 be two vertices in distinct blocks (2-connected maximal subgraph of G). Again, we use the idea mentioned above to construct a sequence $\langle x_1, x_2, \cdots, x_n \rangle$ and the proof follows by using the greedy algorithm for vertex coloring. ■

(•) $\Delta(G) - \chi(G)$ can be arbitraily large.

(•) There are also graphs G such that $\Delta(G) = \chi(G)$, forexample even cycles, non-bipartite 3-regular graphs, say, Petersen graph.

G :

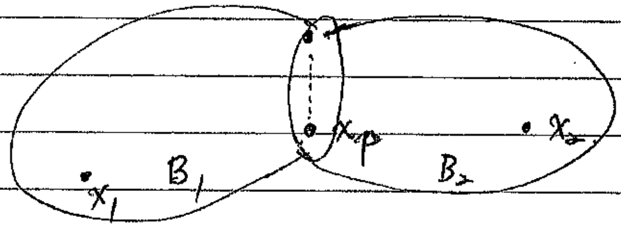
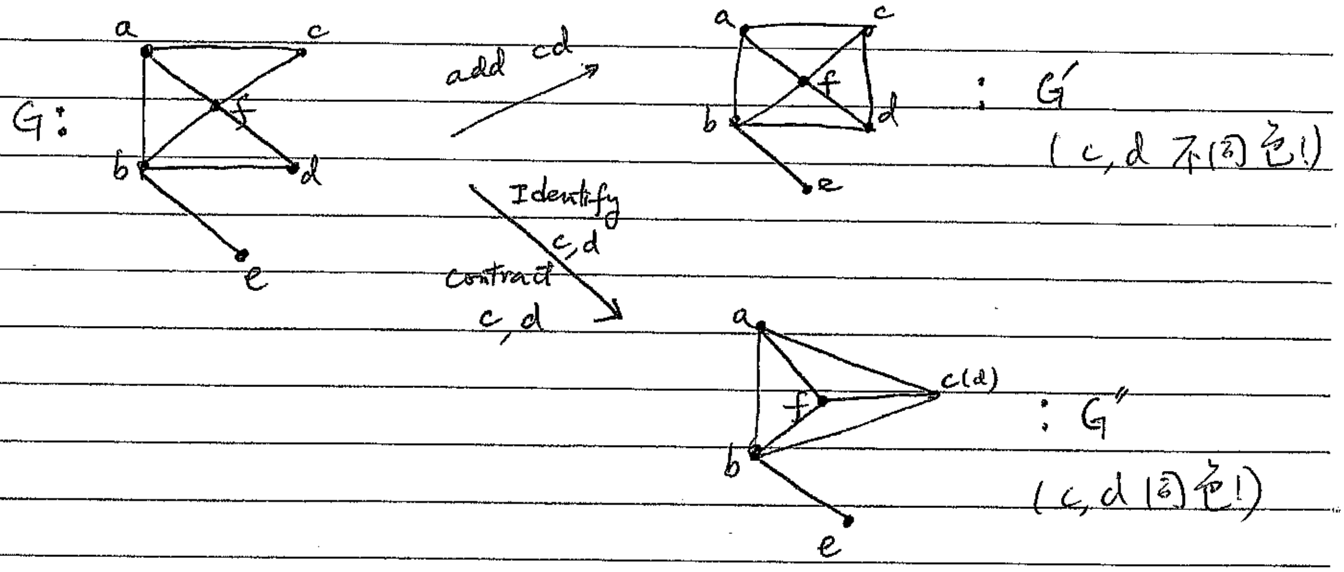


Figure 38. $\kappa(G) = 2$.

Another coloring algorithm (Counting Idea)



Observation A coloring φ of G has two outcomes:

(1) $\varphi(c) \neq \varphi(d)$ and (2) $\varphi(c) = \varphi(d)$.

Theorem 59

Let $p_H(k)$ be the number of distinct k-colorings of H. Then, $p_G(k) = p_{G'}(k) + p_{G''}(k)$ where G' and G'' are graphs obtained from G by adding xy and contracting x and y respectively for $x \not\leftrightarrow_G y$.

($p_G(k)$ is known as the chromatic polynomial of G with k-colorings.)

Proof. The proof follows by the fact that $\varphi(x) = \varphi(y)$ or $\varphi(x) \neq \varphi(y)$ but not both. ∎

(∘) G is k-colorable if and only if $P_G(k) \geq 1$.

(∘) $\chi(G) = \min\{\chi(G'), \chi(G'')\}$.

Theorem 60

Let G be a (p,q)-graph with k components. Then,

$P_G(x) = \sum_{i=0}^{p-k} (-1)^i \cdot a_i \cdot x^{p-i}$ where $a_0 = 1$, $a_1 = q$ and a_i is a positive integer for $0 \leq i \leq p-k$.

Proof. By induction on $p+q$. Clearly, it's true for $p+q=1$. Assume the assertion is true for the cases of smaller $p+q$ and let G be a (p,q)-graph with k components. First, if $m=0$, then $p=k$, so $P_G(x) = x^p$, thus $a_0 = 1$, $a_1 = q = 0$. Now, consider $m \geq 1$. Let uv be an edge of G and $G_0 = G - uv$. By induction,

$P_{G_0}(x) = x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i \cdot x^{p-i}$ where b_i is a non-negative integer for each i. (G_0 has at least k components.) Also,

$P_{G_0''}(x) = x^{p-1} - \sum_{i=2}^{p-k} (-1)^i c_i \cdot x^{p-i}$ where c_i is a positive integer for each i.

Note that $G_0' \cong G$ (adding uv back).

$$p_G(x) = p_{G_0}(x) - p_{G_0'}(x)$$

$$= x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i x^{p-i}$$

$$- x^{p-1} + \sum_{i=2}^{p-k} (-1)^i c_i x^{p-i}$$

$$= x^p - q \cdot x^{p-1} + \sum_{i=2}^{p-k} (-1)^i (b_i + c_i) x^{p-i}$$

$$= x^p - q x^{p-1} + \sum_{i=2}^{p-k} (-1)^i a_i x^{p-i}, \quad a_i > 0 \text{ for each } i. \quad \blacksquare$$

(°) If T is a tree of order p, then for each $k \geq 1$, there are $k \cdot (k-1)^{p-1}$ different k-colorings of T, i.e., $p_T(k) = k \cdot (k-1)^{p-1}$.

$$\| $$

$$k^p - \binom{p-1}{1} \cdot k^{p-1} + \cdots = k^p - (p-1) k^{p-1} + \cdots .$$

Theorem 61 (Nordhaus and Gaddum, 1956)

If G is a graph of order p, then

(1) $2\sqrt{p} \underset{①}{\leq} \chi(G) + \chi(\bar{G}) \underset{①'}{\leq} p+1$, and

(2) $p \underset{②}{\leq} \chi(G) \cdot \chi(\bar{G}) \underset{②'}{\leq} \left[(p+1)/2\right]^2$.

Proof. First, we claim that ②, $\chi(G) \cdot \chi(\bar{G}) \geq p$. For each vertex v of K_p let $\varphi(v) = (\varphi_1(v), \varphi_2(v))$ where φ_1 and φ_2 are chromatic colorings of G and $\bar{G}$ respectively. Since two vertices of K_p are either adjacent in G or $\bar{G}$, all ordered pairs of $v \in V(K_p)$ are distinct. Hence, $\chi(G) \cdot \chi(\bar{G}) \geq p$. (We need p colors for K_p.)

This implies that $\frac{\chi(G)+\chi(\bar{G})}{2} \geq \sqrt{\chi(G)\cdot\chi(\bar{G})} \geq \sqrt{p}$, ① holds.

Now, let $k = \max_{H \leq G} \delta(H)$. We claim that every induced subgraph H' of $\bar{G}$ has minimum degree $p-k-1$, i.e. $\max_{H' \leq \bar{G}} \delta(H') \leq p-k-1$.

Suppose not. Let H'' be an induced subgraph of $\bar{G}$ such that $\delta(H'') = p-k$. Since H'' is an induced subgraph of $\bar{G}$, $H'' \cong \bar{H}$ for some induced subgraph H of G. Let $|H| = h$. Since $\delta(H'') = \delta(\bar{H}) = p-k$, $\deg_H(v) \leq (h-1)-(p-k)$ for each $v \in V(H)$.

Therefore, in G, $\deg_G(v) \le (h-1)-(p-k)+(p-h) = k-1$. On the other hand, $k = \max_{H \preceq G} \delta(H)$ and thus we have an induced subgraph $H''' \preceq G$ such that $\delta(H''') = k$. This implies that $V(H) \cap V(H''') = \phi$. By the fact $|V(H''')| \ge k+1$, $|H| = h \le p-(k+1)$ and thus $|\overline{H}| \le p-(k+1) = p-k-1$. $\delta(\overline{H}) = p-k$ is not possible. This concludes that

$$\max_{H' \preceq \overline{G}} \delta(H') \le p-k-1 \text{ and thus } \chi(\overline{G}) \le p-k-1+1 = p-k$$

(and $\chi(G) \le 1+k$), the proof of ① follows.

Now, for ②, it follows by

$$\sqrt{\chi(G)\cdot\chi(\overline{G})} \le \frac{\chi(G)+\chi(\overline{G})}{2} \le \frac{p+1}{2}. \quad \blacksquare$$

(•) A graph is said to be <u>self-complementary</u> if $G \cong \overline{G}$.

In this situation $\sqrt{p} \le \chi(G) \le \frac{p+1}{2}$. $p=5 \Rightarrow \chi(G) = 3$.

$$\downarrow$$
$$G \cong C_5$$

<u>Problem</u> Let $\omega(G)$ denote the order of a maximum clique, i.e., the order of complete subgraphs with maximum order. Then,

$\chi(G) \ge \omega(G)$. When does the equaily holds?

Clique number of G

(•) A graph G is called <u>perfect</u> if $\chi(H) = \omega(H)$ for each induced subgraph H of G. (*) $\chi(H) - \omega(H)$ can be very large!

<u>Theorem 62</u>

For every integer n, there exists a triangle-free graph G such that $\chi(G) = n$. $(\chi(G) - \omega(G) = n-2.)$

<u>Proof.</u> By induction on n and K_1, K_2, C_5 do have the property respectively for $n = 1, 2$ and 3. Now, assume that H is a triangle-free k-chromatic graph, i.e., $\chi(H) = k$. We construct a graph G based on H such that G is a triangle-free $(k+1)$-chromatic graph.

Let $V(H) = \{v_1, v_2, \dots, v_p\}$ and $V(G) = V(H) \cup \{u_1, u_2, \dots, u_p, u_0\}$.

Let $E(G) = E(H) \cup \{u_0u_i \mid i = 1, 2, \dots, p\} \cup \{u_iv_j \mid v_j \in N_H(v_i)\}$. See Figure 38 for an example when $k = 3$.

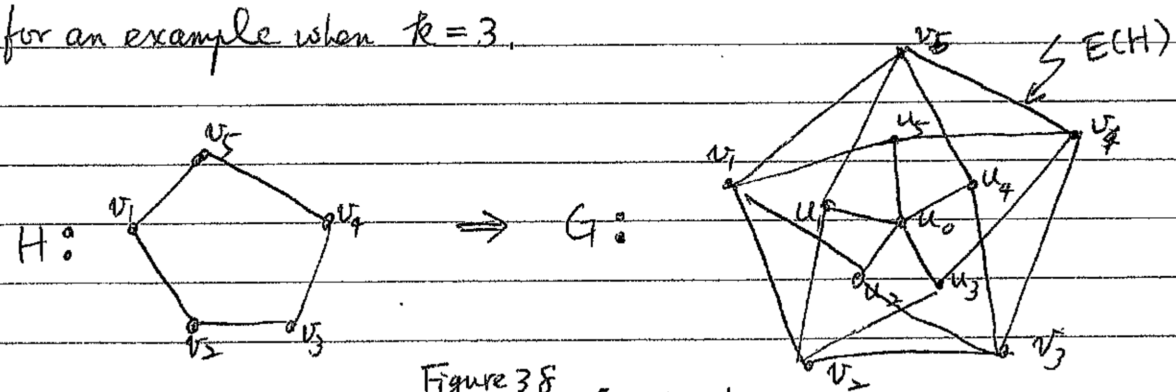


Figure 38 Grötzsch graph

Since $\langle\{u_1, u_2, \cdots, u_p\}\rangle_G$ contains no edges, u_0 is not in any triangle. By assumption, $H \not\supseteq K_3$. So, the only possibility will a triangle consists of u_i, v_j and v_k where $u_i v_j$ and $u_i v_k$ are edges of G. If they form a triangle, then $\langle\{v_i, v_j, v_k\}\rangle_H$ is a triangle in H. Hence, G is triangle-free.

Now, we claim $\chi(G) = k+1$. Let φ be a k-coloring of H. Let $\tilde{\varphi} : V(G) \to \{1, 2, \cdots, k+1\}$ by letting $\tilde{\varphi}(u_i) = \varphi(v_i)$ and $\tilde{\varphi}(u_0) = k+1$. Hence, we have a $(k+1)$-coloring of G, thus $\chi(G) \leq k+1$. On the other hand, we show that $\chi(G) \geq k+1$. Suppose not. Let φ' be a k-coloring of G and the colors used are $1, 2, \cdots, k$. First, we assign u_0 the color k, i.e., $\varphi'(u_0) = k$. So, the colors used for $u_1, u_2, \cdots, u_p$ must be in $\{1, 2, \cdots, k-1\}$. Since $\chi(H) = k$, k occurs somewhere in H, say v_i. (May have more vertices.) Now, we recolor v_i by using $\varphi'(u_i)$. Since u_i is adjacent to every vertex of $N_H(v_i)$, $\varphi'(u_i) \neq \varphi'(v)$ for each $v \in N_H(v_i)$ and thus we have a proper coloring of H using at most $k-1$ colors. (?) $\to\leftarrow$ ■
$(\chi(H) = k)$

(*) This theorem has been extended to a more general result obtained by Erdös and Lovász (1961): For any integers $m, n \geq 2$, there exists an n-chromatic graph whose girth exceeds m. (Theorem 62 is for $m = 3$.)

(For reference)

** Theorem 63 (Lovász, 1972) (Weakly Perfect Graph Theorem)

A graph G is perfect if and only if $\overline{G}$ is perfect.

Note. The proof of this theorem is not too long. But, the proof of next one is long.

*** Theorem 63. (Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas, Annals of Mathematics, 164(2006), 51–229.)

A graph G is perfect if and only if no induced subgraph of G or $\overline{G}$ is an odd cycle of length at least 5.

Proof of Theorem 63

We prove a different version: (Stronger)

A graph G is perfect if and only if $|H| \leq \alpha(H) \cdot \omega(H)$ (1)

for all induced subgraphs H of G. ($\omega(H)$ is the clique number of H.)

($\circ$) In $\overline{G}$, if A is an independent set, then in G, $\langle A \rangle_G$ is a clique

$\langle A \rangle_{\overline{G}}$ (a clique) (A is independent)

($\Rightarrow$) If G is perfect, then for each induced subgraph H, $\chi(H)=\omega(H)$.
(Definition)
Hence, the vertex set of H, $V(H)$, can be partitioned into $\omega(H)$ subsets.

Clearly, each subset has size at most $\alpha(H)$, hence $|H| \le \alpha(H)\cdot\omega(H)$.

($\Leftarrow$) By induction on $|G|$. Assume that every induced subgraph H of G satisfying (1), but <u>G is not perfect.</u> (Every "proper" induced subgraph is perfect.)

Now, let $u \in V(G)$ and consider $G-u$. Let $\omega(G)=\omega$ and $\alpha(G)=\alpha$. By induction, $\chi(G-u)=\omega(G-u)$. If $\omega(G-u)<\omega(G)$, then by coloring u with a new color, we have $\chi(G)\le\omega(G)$. This implies that G is perfect. (We can replace u with an independent set!)

Let K be the vertex set of a clique with ω vertices. Notice that if $u \notin K$, then K meets every color class (independent set) of $G-u$. —(2) But, if $u \in K$, then K meets $\omega-1$ color classes of $G-u$. —(3)

Now, we construct $\alpha\omega+1$ independents in G by the followings.

Let $A_0=\{u_1, u_2, \cdots, u_\alpha\}$ be an independent set of G with α vertices (independence number α).

Starting from $G-u_1$, and then $G-u_2, \cdots, G-u_\alpha$, we have $\alpha\omega$ independent sets: $A_1, A_2, \cdots, A_\omega, A_{\omega+1}, A_{\omega+2}, \cdots, A_{2\omega}, \ldots, A_{\alpha\omega}$. (Each of them contains ω independent sets.)

Observe that $K \cap A_i = \phi$ for all but one $i \in \{0, 1, 2, \cdots, \alpha\omega\}$.

(If $K \cap A_0 = \phi$, then $K \cap A_i \neq \phi$ for all $i \in \{1, 2, \cdots, \alpha\omega\}$ (by (2)). On the other hand, if $K \cap A_0 \neq \phi$, then $|K \cap A_0| = 1$, say $K \cap A_0 = \{u_j\}$.
(Except for u_j, all the other vertices of A_0 are not in K.)
This implies that K meets $\omega - 1$ color classes of $G - u_j$ which implies that in $G-u_j$, there is an independent set A_i such that $K \cap A_i = \phi$.
(By (3).)

Finally, let M and N ((0,1)-matrices) be defined as in Figure 39.

Let $V(G) = \{v_1, v_2, \cdots, v_p\}$. Let $\boxed{K_i \subseteq G - A_i}$ (?) for each $i = 0, 1, \cdots, \alpha\omega$.

$$M: \begin{array}{c|c} & v_1 \; v_2 \; \cdots \; v_p \\ \hline A_0 & \\ A_1 & M(i,j) = 1 \\ \vdots & \text{iff } v_j \in A_i \\ A_{\alpha\omega} & \end{array}_{(\alpha\omega+1)\times p} \qquad N: \begin{array}{c|c} & K_0 \; K_1 \; \cdots \; K_{\alpha\omega} \\ \hline v_1 & \\ v_2 & N(i,j) = 1 \\ \vdots & \text{iff } v_i \in K_j \\ v_p & \end{array}$$

(?) By induction $\chi(G - A_i) = \omega(G - A_i) = \omega(G)$.
otherwise, G is perfect.

Since $M \cdot N = J_{(\alpha\omega+1)\times(\alpha\omega+1)} - I_{\alpha\omega+1}$ is non-singular, the rank of $M \cdot N$ is $\alpha\omega+1$ which is larger than $p = |G|$, a contradiction to the assumption when $H \cong G$. Hence, the proof follows. ∎

✓ Theorem 64 If G is a connected planar graph, then $\chi(G) \leq 5$.

Proof. By induction on $|G|$. By Theorem 57, it suffices to consider an induced subgraph H whose minimum degree is 5.

Let $v \in V(H)$ such that $\deg_H(v) = 5$. By induction, $\chi(H-v) \leq 5$. Let φ be a 5-coloring of H and we consider the colors assigned on $N_H(v)$. Let them be $\varphi(v_1), \varphi(v_2), \cdots, \varphi(v_5)$. Clearly, if any two of them are of the same color, then there is a color for v such that we have a proper 5-coloring of H. So, assume that $\varphi(v_i) = i$, $i = 1,2,3,4,5$ and the vertices are in clockwise order, see Figure 40.

Now, consider the induced subgraph $H_{1,3} = \langle \varphi^{-1}(1) \cup \varphi^{-1}(3) \rangle_H$. If v_1 and v_3 are in distinct components, then by changing the colors 1 and 3 in the component which contains v_1, we obtain a new coloring such that $\varphi(v_1) = 3$ and $\varphi(v_3) = 3$. Hence, 3 is available for v.

On the other hand, there exists a path P connecting v_1 and v_3. Hence, $v - v_1 - P - v_3 - v$ is a cycle such that v_2 and v_4 are in different regions. By a similar argument, we may change the color of v_2 to 4. Then, 2 is available for v. ■

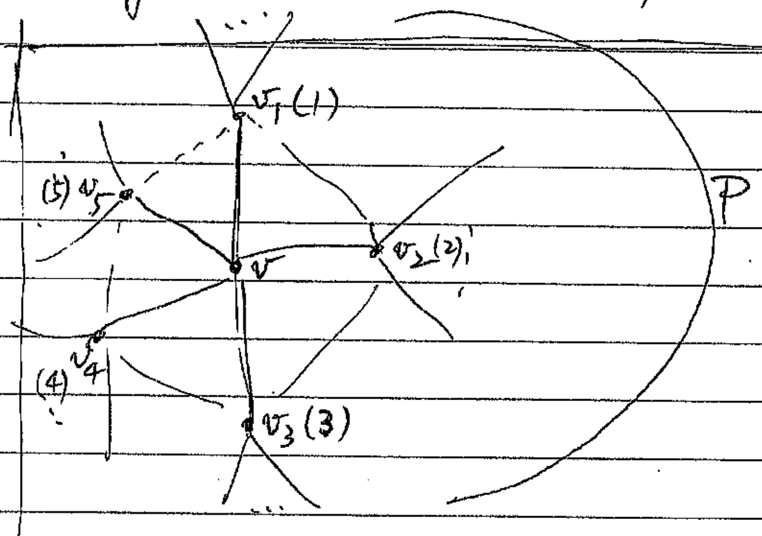


Figure 40

(~~***~~) Theorem (4CT) Every planar graph is 4-colorable.

The most recent proof was obtained by N. Robertson, D.P. Sanders, P.D. Seymour and R. Thomas (1996): A new proof of the 4CT, Electron. Res. Announc. A.M.S. 2, 17-25.

The first proof was obtained in ~~1977~~ 1976, by K. Appel and W. Haken.

11-18

Discharging Method (Mainly on planar graphs)

Step 1. Charging

Give charges to each vertex and each face.
(Count the total charge.)

Step 2. Discharging

Discharge those vertices or faces with positive charges.
(The total charge remains the same.)

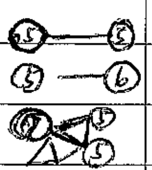
(o) Step 1 and Step 2 are called charging and discharging rules respectively.

(*) Use the charging (discharging) results to find the structure of graphs.

For example (Theorem)

If a planar graph G (maximal) has $\delta(G)=5$, then it either has an edge with endpoints (endvertices) of degree 5 or one with endpoints (endvertices) of degree 5 and 6. (light edge)

Proof. Charging (Step 1): vertex v, $6-\deg(v)$
face f, $6-2|f|$. $(=0$, maximal$)$



Discharging (Step): discharge $\frac{1}{5}$ to its neighbors if the vertex is of degree 5.

$\Rightarrow$ Each vertex with positive charge (final) (of degree ≤ 7) is adjacent to an endpoint of a light edge.

Per-Duet

WIKIPEDIA

Discharging method (discrete mathematics)

The **discharging method** is a technique used to prove lemmas in structural graph theory. Discharging is most well known for its central role in the proof of the four color theorem. The discharging method is used to prove that every graph in a certain class contains some subgraph from a specified list. The presence of the desired subgraph is then often used to prove a coloring result.

Most commonly, discharging is applied to planar graphs. Initially, a *charge* is assigned to each face and each vertex of the graph. The charges are assigned so that they sum to a small positive number. During the *Discharging Phase* the charge at each face or vertex may be redistributed to nearby faces and vertices, as required by a set of discharging rules. However, each discharging rule maintains the sum of the charges. The rules are designed so that after the discharging phase each face or vertex with positive charge lies in one of the desired subgraphs. Since the sum of the charges is positive, some face or vertex must have a positive charge. Many discharging arguments use one of a few standard initial charge functions (these are listed below). Successful application of the discharging method requires creative design of discharging rules.

An example

In 1904, Wernicke introduced the discharging method to prove the following theorem, which was part of an attempt to prove the four color theorem.

Theorem: If a planar graph has minimum degree 5, then it either has an edge with endpoints both of degree 5 or one with endpoints of degrees 5 and 6.

Proof: We use V, F, and E to denote the sets of vertices, faces, and edges, respectively. We call an edge *light* if its endpoints are both of degree 5 or are of degrees 5 and 6. Embed the graph in the plane. To prove the theorem, it is sufficient to only consider planar triangulations (because, if it holds on a triangulation, when removing nodes to return to the original graph, neither node on either side of the desired edge can be removed without reducing the minimum degree of the graph below 5). We arbitrarily add edges to the graph until it is a triangulation. Since the original graph had minimum degree 5, each endpoint of a new edge has degree at least 6. So, none of the new edges are light. Thus, if the triangulation contains a light edge, then that edge must have been in the original graph.

We give the charge $6 - d(v)$ to each vertex v and the charge $6 - 2d(f)$ to each face f, where $d(x)$ denotes the degree of a vertex and the length of a face. (Since the graph is a triangulation, the charge on each face is 0.) Recall that the sum of all the degrees in the graph is equal to twice the number of edges; similarly, the sum of all the face lengths equals twice the number of edges. Using Euler's Formula, it's easy to see that the sum of all the charges is 12:

$$\begin{aligned}\sum_{f\in F} 6 - 2d(f) + \sum_{v\in V} 6 - d(v) &= \\ 6|F| - 2(2|E|) + 6|V| - 2|E| &= \\ 6(|F| - |E| + |V|) &= \quad 12.\end{aligned}$$

We use only a single discharging rule:

- Each degree 5 vertex gives a charge of 1/5 to each neighbor.

We consider which vertices could have positive final charge. The only vertices with positive initial charge are vertices of degree 5. Each degree 5 vertex gives a charge of 1/5 to each neighbor. So, each vertex is given a total charge of at most $d(v)/5$. The initial charge of each vertex v is $6 - d(v)$. So, the final charge of each vertex is at most $6 - 4d(v)/5$. Hence, a vertex can only have positive final charge if it has degree at most 7. Now we show that each vertex with positive final charge is adjacent to an endpoint of a light edge.

If a vertex v has degree 5 or 6 and has positive final charge, then v received charge from an adjacent degree 5 vertex u, so edge uv is light. If a vertex v has degree 7 and has positive final charge, then v received charge from at least 6 adjacent degree 5 vertices. Since the graph is a triangulation, the vertices adjacent to v must form a cycle, and since it has only degree 7, the degree 5 neighbors cannot be all separated by vertices of higher degree; at least two of the degree 5 neighbors of v must be adjacent to each other on this cycle. This yields the light edge.

References

- Appel, Kenneth; Haken, Wolfgang (1977), "Every planar map is four colorable. I. Discharging", *Illinois Journal of Mathematics*, **21**: 429–490, doi:10.1215/ijm/1256049011 (https://doi.org/10.1215%2Fijm%2F1256049011).
- Appel, Kenneth; Haken, Wolfgang (1977), "Every planar map is four colorable. II. Reducibility", *Illinois Journal of Mathematics*, **21**: 491–567, doi:10.1215/ijm/1256049012 (https://doi.org/10.1215%2Fijm%2F1256049012).
- Hliněný, Petr (2000), *Discharging technique in practice* (http://kam.mff.cuni.cz/~kamserie/serie/clanky/2000/s475.ps). (Lecture text for Spring School on Combinatorics).
- Robertson, Neil; Sanders, Daniel P.; Seymour, Paul; Thomas, Robin (1997), "The four-color theorem", *Journal of Combinatorial Theory, Series B*, **70**: 2–44, doi:10.1006/jctb.1997.1750 (https://doi.org/10.1006%2Fjctb.1997.1750).
- Wernicke, P. (1904), "Über den kartographischen Vierfarbensatz" (https://zenodo.org/record/1428228/files/article.pdf) (PDF), *Math. Ann.* (in German), **58** (3): 413–426, doi:10.1007/bf01444968 (https://doi.org/10.1007%2Fbf01444968).

(*) The following theorem is not working for $n=0$.

Theorem 65 (The Heawood Map Coloring Theorem)

For every positive integer n, $\chi(S_n) = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$.

($\chi(S_n)$: the maximum chromatic number among all graphs that can be embedded on S_n.)

Proof. (Outline)

The upper bound $\chi(S_n) \le \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$ was obtained by Heffter in 1890. At that time, he claimed that it's an equality. But, unfortunately, the correct proof came out many years later by the effort of considering the embedding of K_p since for sure K_p needs p colors.

So, define $p = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$. It follows that $n \ge (p-3)(p-4)/12$ and thus $n \ge \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$. By the fact $\gamma(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$, $\chi(S_n) \ge p = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$. ∎