

(\*) Topological Graph Theory studies the "drawing" of a graph on a surface.

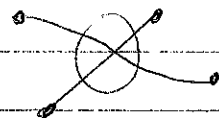
(\*\*) A proper drawing on a surface of a graph  $G$  with  $p$  vertices and  $q$  edges follows the rules:

(1). There are  $p$  points on the surface which corresponds to the set of vertices in  $G$ ; and

(2). There are  $q$  curves joining points defined above which correspond to the set of edges and they are pairwise

disjoint except possibly for the endpoints.

(3). Two edges have at most one crossing.



2-manifold: A connected topological space in which every

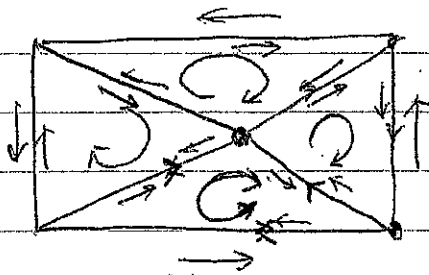
point has a neighborhood homeomorphic to the open unit disk defined on  $\mathbb{R}^2$ .

Bounded subspace: A subspace  $M$  of  $\mathbb{R}^3$  is bounded if  $\exists K \in \mathbb{R}^+$

such that  $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 = K\}$ .

Closed:  $M$  is closed if its boundary  $\partial M$  coincides with  $M$ .

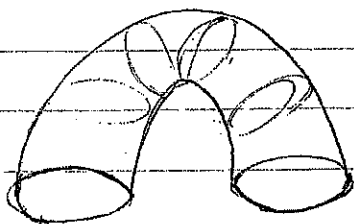
**Orientable** :  $M$  is orientable if for every simple closed curve  $C$  on  $M$ , a clockwise sense of rotation is preserved once around  $C$ . Otherwise,  $M$  is non-orientable.



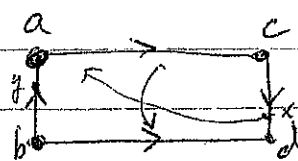
**Orientable**

**Surface** : A surface  $S_g$  is a compact "orientable" 2-manifold that may be thought of as a sphere on which has been placed (inserted) a number  $g$  of "handles" (holes),

**Non-orientable** : A surface obtained by adding to cross-caps to a sphere ( $S_0$ ) is a non-orientable surface  $N_g$ .



Handle

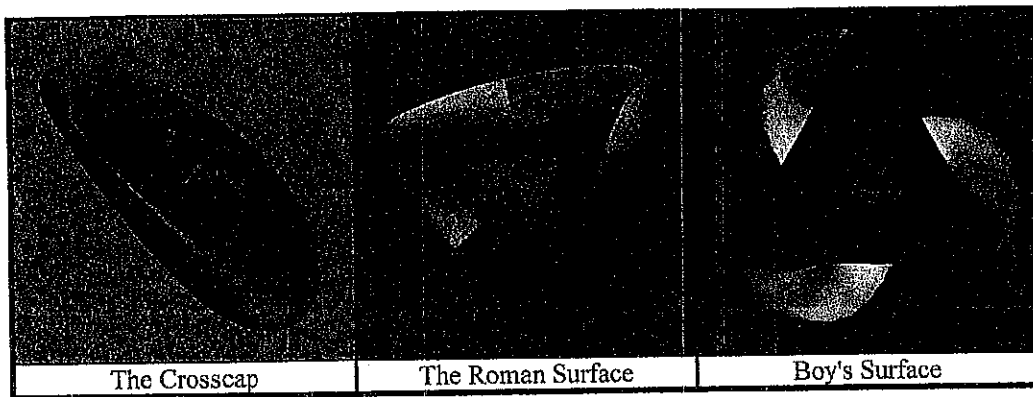


Möbius band

$$a \rightarrow d, c \rightarrow b, x \rightarrow y$$



(\*) Adding a crosscap: Attach the boundary of a Möbius band to a cycle on  $S_0$ .



### Definition (Embeddable)

A  $(p, q)$ -graph  $G$  is said to be embeddable on a surface if it is possible to draw  $G$  properly on the surface.  
(drawing without crossings)

### Definition (Planar graph)

A graph is planar if it can be embedded in the plane, equivalently, embedded on the sphere.

### Definition (2-cell embedding)

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An embedding in which for each region all curves in that region can be continuously deformed or contracted in that region to a single point.

A 2-cell is topologically homeomorphic to  $\mathbb{R}^2$ . An embedding of  $G$  on a surface is a 2-cell embedding of  $G$  if all the regions so determined are 2-cells.

The following figure shows embeddings of  $K_{3,3}$  on  $S_1$  and  $S_2$  respectively.  $S_k$  is a surface obtained by attaching  $k$  handles.

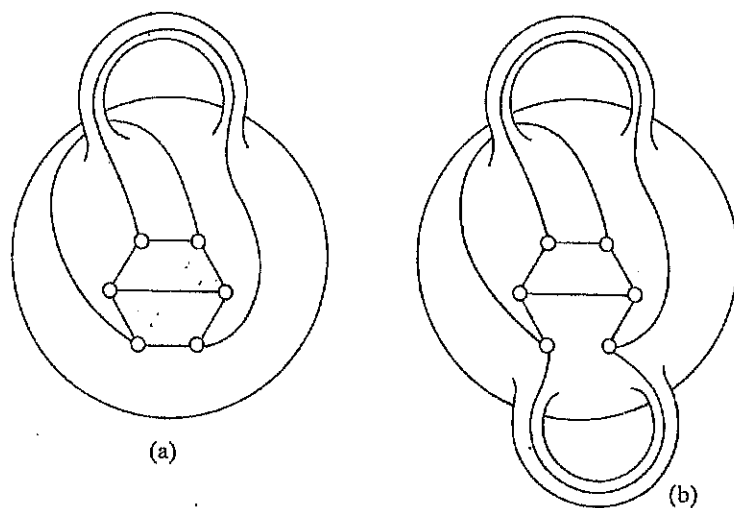


Figure 3.1. Embeddings of  $K_{3,3}$  on surfaces  $S_1, S_2$ .

(\*)  $S_0$  : Sphere

(\*)  $N_0 \cong S_0$ ,  $N_h$  : Attach  $h$  crosscaps to  $N_0(S_0)$ .

## Definition (Genus)

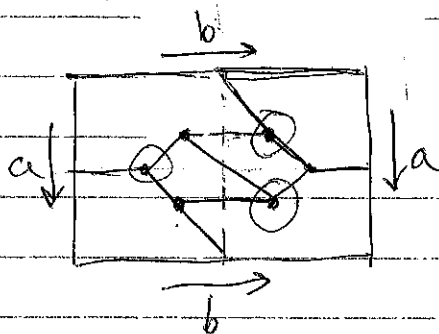
The number of handles (resp. crosscaps) <sup>(on a surface)</sup> is referred to as <sup>orientable</sup> genus of the surface (resp. non-orientable surface). We use

$\gamma(G)$  (respectively  $\tilde{\gamma}(G)$ ) to denote the smallest genus of all <sup>orientable</sup> surfaces (resp. non-orientable surfaces) on which  $G$  can be embedded.

(\*) If  $G$  is a planar graph, then  $\gamma(G)$  (so is  $\tilde{\gamma}(G)$ ) is equal to zero. But,  $G$  can be embedded on a surface with genus larger than "0".

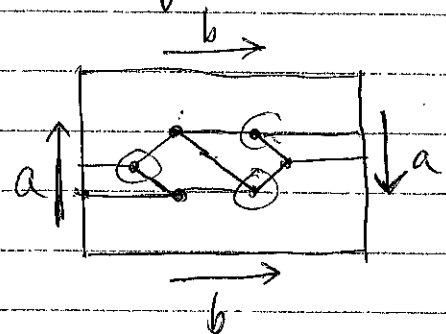
(\*\*) Given a graph  $G$ , determining  $\gamma(G)$  is a difficult problem.

Orientable embedding  
of  $K_{3,3}$  ( $S_1$ )



$$bab^{-1}a^{-1}$$

Non-orientable embedding  
of  $K_{3,3}$  (Möbius band)



$$bab^{-1}a$$

Theorem 46 (Euler)

Let  $G$  be a <sup>connected</sup> planar graph with  $p$  vertices,  $q$  edges and  $f$  faces (regions). Then  $p - q + f = 2$ .

Proof. By induction on  $q$ . Since  $G$  is connected,  $G$  has at least  $p-1$  edges. (?) If  $G$  has  $p-1$  edges and  $G$  is connected,

then  $G$  is a tree which contains no cycles. This implies that

$f = 1$  and thus  $p - (p-1) + 1 = 2$ . The assertion is true

for "minimal" graphs. Let the hypothesis is true for  $\|G\| \geq p-1$ .

Now, consider  $G$  with  $k+1$  edges. Clearly,  $G$  contains a cycle.

Let  $e$  be a cycle edge. Since  $G$  is a connected planar

graph,  $G-e$  is also a connected planar graph. Moreover,

$\|G-e\| = k$  and  $G-e$  has  $k$  edges and  $f'-1$  faces. By

induction  $p - k + f' - 1 = 2$  and thus

$p - (k+1) + f' = 2$ . This concludes the proof.  $\square$

Theorem 49 If  $G$  is a planar graph, then  $\|G\| = 3|G| - 6$ .

Proof By observation, if  $G$  has maximum size, then each region of  $G$  is a triangle. Since each edge of  $G$  is in the boundary of exact two regions,  $3 \cdot f = 2 \cdot q$  where  $f$  is the number of regions and  $q$  is the size of  $G$ , i.e.,  $q = \|G\|$ . Now, by Euler's formula  $p - q + f = 2$  equivalently

$$|G| - \|G\| + \frac{2}{3}\|G\| = 2$$

$$\Rightarrow 3|G| - 6 = \|G\|. \quad (G \text{ is a maximal planar graph!})$$

Corollary If  $G$  is a planar graph, then  $\|G\| \leq 3|G| - 6$ .

(\*)  $K_5$  is not planar.

Corollary In any <sup>planar</sup> graph, there exists at least one vertex of degree smaller than 6.

Corollary The degree sum of a planar graph is at most  $6|G| - 12$ .

This corollary is very useful.

(\*) We can give a more accurate estimation of the above corollary

(\*) If  $G$  is a planar graph with girth  $g(G) \geq 4$ , then  $q \leq 2p - 4$ .

$$4f \leq 2q, \quad q \geq 2f, \quad p - q + f = 2 \leq p - q + \frac{q}{2} \Rightarrow \frac{q}{2} \leq p - 2 \Rightarrow q \leq 2p - 4.$$

$\Rightarrow K_{3,3}$  is not planar.

Theorem 48

Let  $G$  be a maximal planar graph (triangulated) of order  $p$ , and let  $p_i$  denote the number of vertices of degree  $i$  in  $G$  for  $i=3, 4, \dots, \Delta(G)=d$ . Then

$$3p_3 + 2p_4 + p_5 = p_7 + 2p_8 + \dots + (d-6)p_d + 12.$$

Proof. Since  $p = \sum_{i=3}^d p_i$  and  $2q = \sum_{i=3}^d i p_i$ , we have

$$\sum_{i=3}^d i p_i = 2(3p-6) = 6 \sum_{i=3}^d p_i - 12.$$

This implies the conclusion. ▀

Theorem 49

There are exactly five regular polyhedra.

Proof. Notice that a regular polyhedron is a polyhedron whose faces (regions) are bounded by congruent regular polygons and whose polyhedral angles are congruent.

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First, we convert a polyhedron into a regular planar graph.

(See Figure 3.2.) Let the number of vertices, edges and faces be



By Theorem 46,  $p - q + f = 2$ . Hence,

$$-8 = 4q - 4p - 4f = 2q + 2q - 4p - 4f$$

$$= \sum_{i \geq 3} i f_i + \sum_{i \geq 3} i p_i - 4 \sum_{i \geq 3} p_i - 4 \sum_{i \geq 3} f_i \quad (f_i: \# \text{ of } i\text{-face})$$

$$= \sum_{i \geq 3} (i-4) f_i + \sum_{i \geq 3} (i-4) p_i.$$

Since the polyhedron is regular, all degrees and face sizes are the same, let them be  $k$  and  $h$  respectively. Therefore,

$$-8 = (h-4) f_h + (k-4) p_k.$$

By the fact that every planar graph contains a vertex of degree less than six, we only have nine cases to consider:

$$3 \leq h \leq 5 \text{ and } 3 \leq k \leq 5.$$

From direct checking, only 5 cases are possible, namely,

- (1)  $f_3 = p_3 = 4$  (Tetrahedron) 四面体
- (2)  $f_3 = 8$  and  $p_3 = 6$  (Octahedron) 八面体
- (3)  $f_3 = 20$  and  $p_3 = 12$  (Icosahedron) 二十面体
- (4)  $f_4 = 6$  and  $p_3 = 8$  (Cube) 六面体
- (5)  $f_5 = 12$  and  $p_3 = 20$  (Dodecahedron) 十二面体. ▣

See Figure 32 for regular polyhedra.

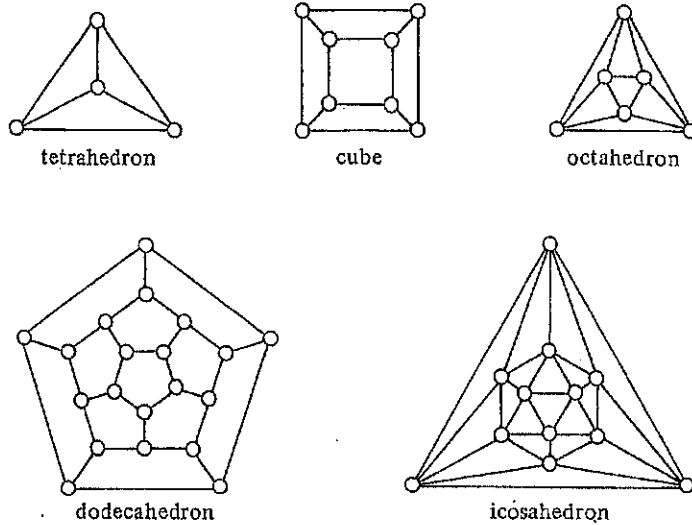


Figure 32 The graphs of the regular polyhedra

Theorem 50 (Fáry, Wagner,  
(1948) (1936))

A planar graph  $G$  can be embedded in the plane so that each edge is a straight line segment.

Proof. The proof is by induction on the order of  $G$ . It suffices to prove the case when  $G$  is a connected <sup>maximal</sup> planar graph.

Clearly, it is true for small orders. Assume the hypothesis is true for order  $k$  and let  $G$  be a connected maximal planar graph of order  $k+1$ .

Since  $G$  is maximal,  $3 \leq \delta(G) \leq 5$ .

Case 1.  $\delta(G) = 3$

Let  $v_0 \in V(G)$  such that  $\deg_G(v_0) = 3$  and  $v_0$  is adjacent to  $v_1, v_2$  and  $v_3$ . Since  $G$  is maximal,  $\langle \{v_1, v_2, v_3\} \rangle_G \cong K_3$ . This implies that  $G - v_0$  is also a <sup>maximal</sup> planar graph. By induction  $G - v_0$  has a straight line segment embedding. Now, put  $v_0$  back to the graph  $G - v_0$  such that  $v_0$  is inside the region bounded by  $\langle \{v_1, v_2, v_3\} \rangle$  and connect  $v_0$  to the three vertices by straight line segment. This concludes the proof of this case.

Case 2.  $\delta(G) = 4$

The proof follows by a similar process by letting  $N(v_0) = \{v_1, v_2, v_3, v_4\}$ . Now,  $G - v_0 + v_1, v_3$  is a maximal planar graph and thus it has a straight line segment embedding. The proof follows by placing  $v_0$  back to  $G - v_0 + v_1, v_3 = v_1, v_3$ . By considering the drawing of the embedding (Figure 33(a)), we are able to put

segment.

Case 3.  $\delta(G) = 5$ .

Again, we use the same technique and the drawing can be seen

in Figure 33(b).

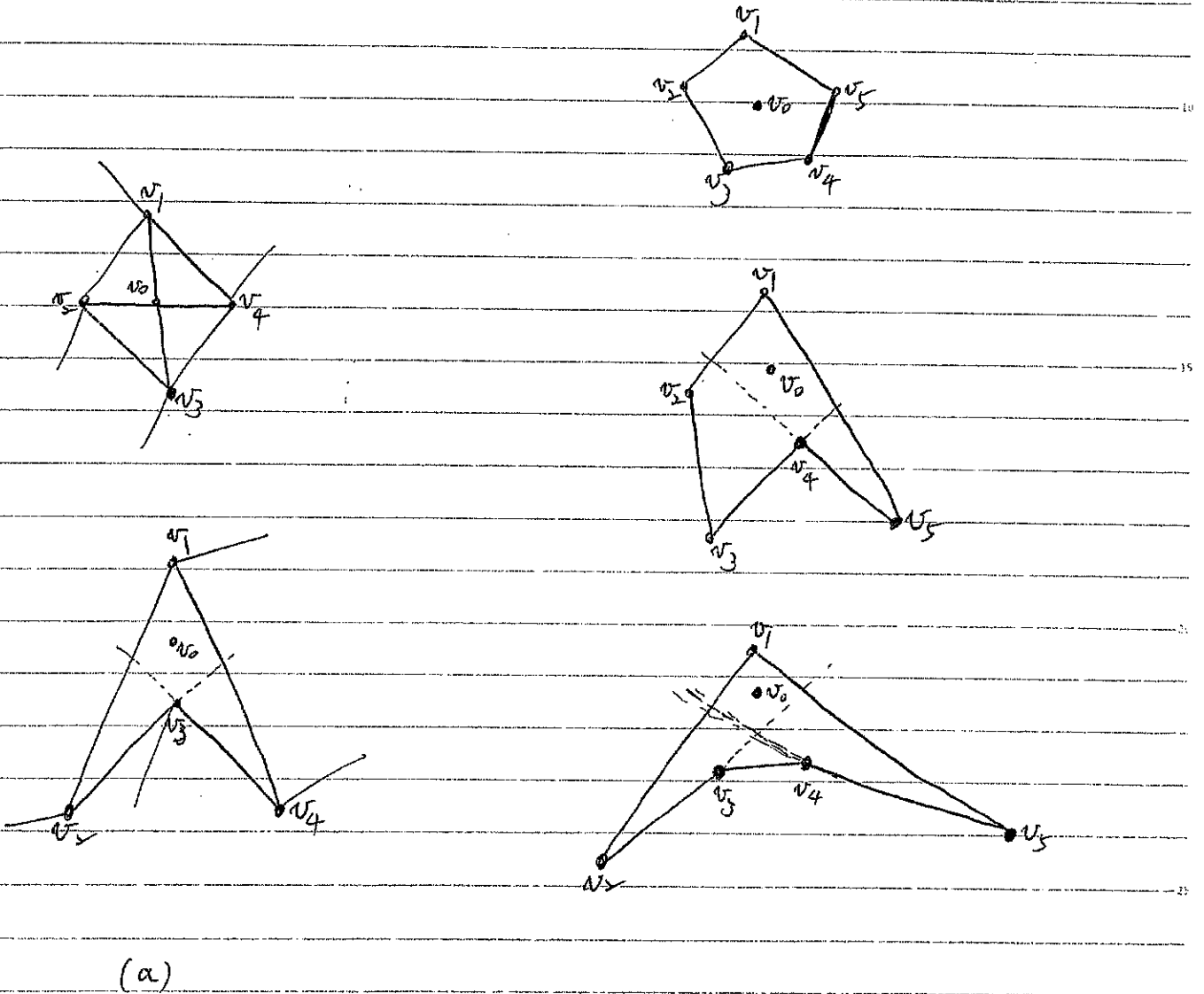
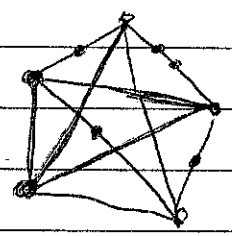


Figure 33 Location of  $v_0$ .

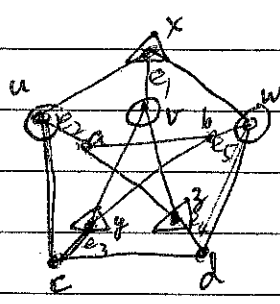
# Characterization of planar graphs

(\*)  $K_5$  and  $K_{3,3}$  are not planar graphs so are their homeomorphic graphs.



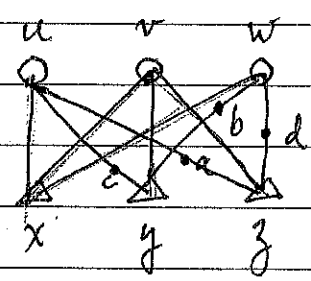
Subdivision of edges

(\*) A graph which contains  $K_5, K_{3,3}$  as minors.



$$\left( \left( \left( \left( G/e_1 \right) / e_2 \right) / e_3 \right) / e_4 \right) / e_5$$

$$G / \{e_1, e_2, e_3, e_4, e_5\} \rightarrow K_5$$



delete ab, cd.

## Theorem (Kuratowski, 1930)

A graph is planar if and only if it does not contain a subdivision of  $K_5$  or  $K_{3,3}$ .

Proof. (Refer to books of Graph Theory.)

(\*) The following theorem considers pseudographs, i.e., loops and multiedges are allowed.

### Theorem 51 (Euler-Poincaré Theorem)

Let  $G$  be a  $(p, q)$ -pseudograph which has a 2-cell embedding on  $S_n$ . Then,  $p - q + f = 2 - 2n$  where  $f$  is the number of faces in the embedding.

Proof. By induction on  $n$  and it's true when  $n = 0$  (by

Euler's planar graph formula). Assume that the assertion is

true when  $n = k \geq 0$  and  $G$  is a  $(p, q)$ -pseudograph which

has a 2-cell embedding on  $S_{k+1}$ . Since  $k+1 \geq 1$ , there exists

a handle in the embedding, see Figure 34(a). It suffices to

consider the embedding such that there exists at least one

edge which passes through the handle (on the surface). Note that

if we can pull back an edge without passing the handle, then

pull it back, see Figure 34(b). Now, we apply the idea of

"cut and past" to obtain a 2-cell embedding<sub>n</sub> of  $\tilde{G}$  on  $S_k$ .

By using a circle around the handle, we can cut the handle through the circle and obtain  $\tilde{G}$ , see Figure 34(c). As a consequence, if there are  $t$  edges passing through the handle the graph  $\tilde{G}$  is embedded in  $S_p$ . Then  $|\tilde{G}| = p + 2t$ ,  $\|\tilde{G}\| = q + 3t$ , and the embedding in  $S_p$  has  $f + t + 2$  faces. Hence,

$$(p + 2t) - (q + 3t) + (f + t + 2) = 2 - 2k.$$

This implies that  $p - q + f = 2 - 2(k+1)$ . ■

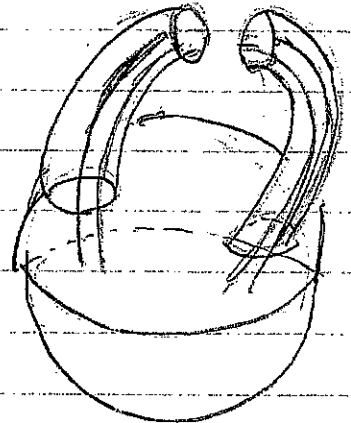
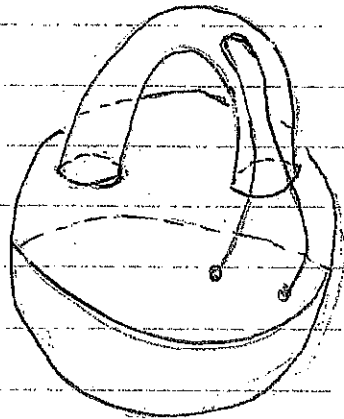
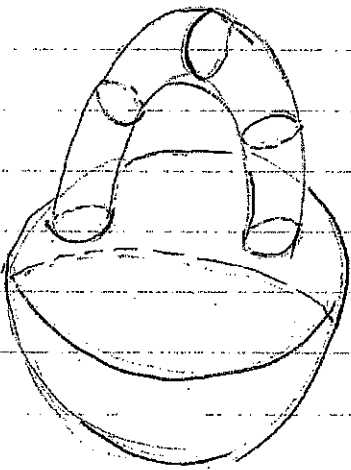


Figure 34 (a)

Figure 34 (b), pull edges back

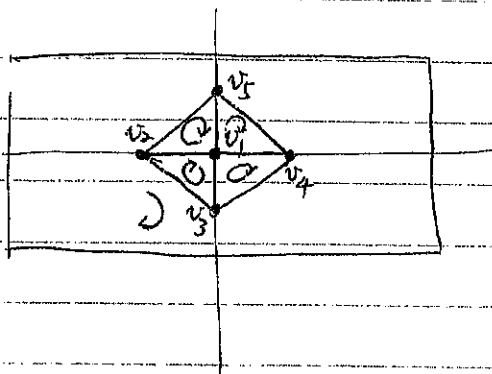
Figure 34 (c)

## For reference

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(\*) How to find a 2-cell embedding on  $S_g$ ?



The above figure provides an embedding of  $K_5$  on  $S_1$ , their regions:

are  $((v_1, v_2, v_5))$ ,  $((v_1, v_3, v_5))$ ,  $((v_1, v_4, v_5))$ , and

$((v_2, v_3, v_5, v_2, v_4, v_5, v_3, v_4))$ .

(\*) Each arc of  $D_5$  occurs exactly once in a region (face).

(\*\*) By considering each vertex, we observe that there are five permutations to determine this embedding.

$$\pi_1 = (2, 3, 4, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix} \text{ (Defined on } v_1 \text{)}$$

$$\pi_2 = (3, 1, 5, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{pmatrix} \text{ (Defined on } v_2 \text{)}$$

$$\pi_3 = (4, 1, 2, 5)$$

$$\pi_4 = (3, 2, 5, 1)$$

$$\pi_5 = (1, 4, 3, 2)$$



On the other hand, if we are given five permutations one from each vertex, then we have a 2-cell embedding.

For example,  $\pi_1 = (3, 2, 4, 5)$ ,  $\pi_2 = (3, 1, 5, 4)$ ,  $\pi_3 = (4, 1, 2, 5)$ ,  $\pi_4 = (3, 2, 5, 1)$ , and  $\pi_5 = (1, 4, 3, 2)$ .

Now, if we start from the arc  $(v_1, v_2)$ , then the permutation of  $\pi_2$  will give the next vertex of the oriented 2-cell containing

$$(1, 2), \quad v_1 - v_2 - v_{\pi_2(1)} \Rightarrow v_1 - v_2 - v_5 - v_{\pi_2(2)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_{\pi_2(5)}$$

$$\Rightarrow v_1 - v_2 - v_5 - v_1 - v_3 \Rightarrow \dots \Rightarrow (v_1, v_2, v_5, v_1, v_3, v_2, v_1, v_4, v_3).$$

$$(v_{\pi_3(4)} = v_1 \text{ and } v_{\pi_1(3)} = v_2)$$

(\*) Therefore, given a  $(p, q)$ -graph  $G$ , we can define  $p$  permutations for  $p$  vertices  $v_i$  such that each permutation is a cycle using the  $q$  vertices in  $N_G(v_i)$ . Then, a 2-cell embedding will be obtained.

(Heffter)

Theorem 52 (The Rotational Embedding Scheme)

Let  $G$  be a nontrivial connected graph with  $V(G) = \{v_1, v_2, \dots, v_p\}$ .

For each 2-cell embedding of  $G$  on a surface, there exists a

unique  $p$ -tuple  $(\pi_1, \pi_2, \dots, \pi_p)$  where for  $i = 1, 2, \dots, p$ ,  $\pi_i$

is a cyclic permutation of  $V(i)$  that describes the subscripts of the vertices in  $N_G(v_i)$  in counterclockwise order about  $v_i$ .

Conversely, for each  $p$ -tuple  $(\pi_1, \pi_2, \dots, \pi_p)$ , there exists a 2-cell embedding of  $G$  on some surface such that for  $i = 1, 2, \dots, p$ , the subscripts of the vertices adjacent to  $v_i$  are in counterclockwise order about  $v_i$  are given by  $\pi_i$ . Moreover,

the set  $\{\pi_1, \pi_2, \dots, \pi_p\}$  induces a mapping  $\pi$  such that

$$\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$$

for each pair of adjacent vertices  $v_i$  and  $v_j$ ,  $1 \leq i \neq j \leq p$ .

Proof. The scheme was first observed and used by Dyck (1888) and Heffter (1891). A formalized version was obtained later in 1960 by Edmonds.

The main idea has been mentioned before the statement of Theorem 52. The genus of surface can be obtained after the number of faces (regions) has been determined.  $\square$

Theorem 53  $\gamma(K_{2m,2n}) = (m-1)(n-1)$ ,  $m \leq n$ .

Proof. For convenience, let  $K_{2m,2n} = (A_0, A_2)$  where

$A_0 = \{a_1, a_3, \dots, a_{4m-1}\}$  and  $A_2 = \{a_2, a_4, \dots, a_{4n}\}$ . Note that  $m$  may

not be equal to  $n$ . See Figure 35 for the case  $2m=6$  and  $2n=8$ .

Now, let

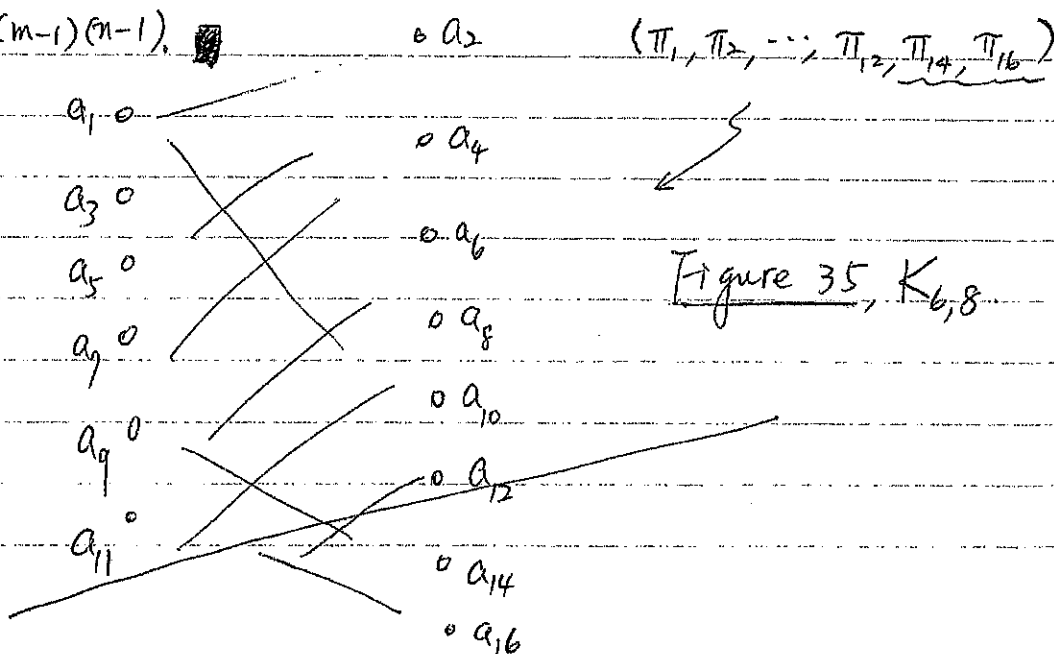
$$\left\{ \begin{array}{l} \pi_1 = \pi_5 = \dots = \pi_{4m-3} = (2 \ 4 \ 6 \ \dots \ 4m); \\ \pi_3 = \pi_7 = \dots = \pi_{4m-1} = (4m \ 4m-2 \ \dots \ 6 \ 4 \ 2); \\ \pi_2 = \pi_6 = \dots = \pi_{4m-2} = (1 \ 3 \ 5 \ \dots \ 4m-1); \text{ and} \\ \pi_4 = \pi_8 = \dots = \pi_{4m} = (4m-1, 4m-3 \ \dots \ 5 \ 3 \ 1). \end{array} \right.$$

We may check that this embedding does have  $2mn$  regions

each of them is bounded by a 4-cycle. Hence, by Theorem 51,

$$2\gamma(G) = 2 - p + q - r = 2 - (2m+2n) + 4mn - 2mn = 2mn - 2m - 2n + 2.$$

Hence,  $\gamma(G) = (m-1)(n-1)$ .



(oo) There are  $\prod_{i=1}^p (\deg_G(v_i) - 1)!$   $p$ -tuples of  $(\pi_1, \pi_2, \dots, \pi_p)$ .

(ooo) The 2-cell embedding with the largest number of faces gives the genus of  $G$ .

(\*) The 2-cell embedding with the smallest number of faces gives the "maximum" genus of  $G$ , denoted by  $\gamma_M(G)$ .

(\*\*) Finding  $\gamma(G)$  is a very difficult problem in general.

(\*\*\*) Finding  $\gamma_M(G)$  is comparatively easier.

Theorem 54. Let  $cr(G)$  denote the crossing number of  $G$ .

Then,  $cr(K_5) = cr(K_{3,3}) = 1$  and  $cr(K_6) = 3$ .

Proof. Since  $\gamma(K_5) = \gamma(K_{3,3}) = 1$ , the proof follows by a drawing with "1" crossing number. Now, we consider  $K_6$ . By Figure 35,

$cr(K_6) \leq 3$ . Let  $cr(K_6) = k$ . Then, we may convert the

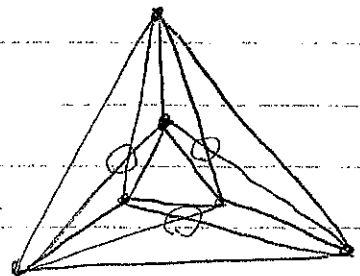
crossings into new vertices. Hence, we have

a <sup>planar</sup> graph  $G$  (obtained from above):  $|G| = 6 + k$

and  $\|G\| = 15 + 2k$ . By the fact

$15 + 2k \leq 3(6 + k) - 6$ , we have  $k \geq 3$ . ▀

Figure 35



Theorem 55  $cr(K_9) = 36$ .

Proof. For the upper bound, it suffices to give a drawing which has exactly 36 crossings. But, it is very technical to show the lower bound. Here, we provide a drawing for  $cr(K_9) \leq 36$ , see Figure 36.

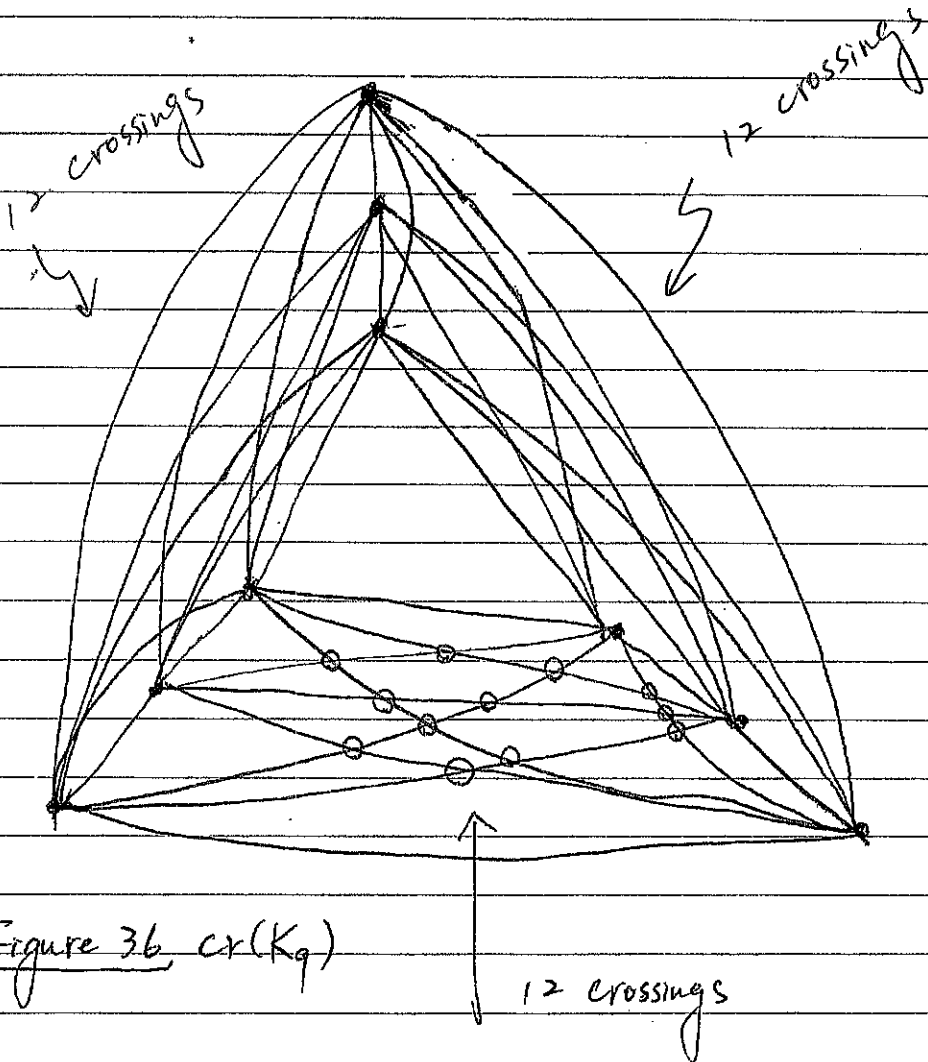


Figure 36,  $cr(K_9)$

Conjecture  $cr(K_p) = \frac{1}{4} \lfloor \frac{p}{2} \rfloor \lfloor \frac{p-1}{2} \rfloor \lfloor \frac{p-2}{2} \rfloor \lfloor \frac{p-3}{2} \rfloor$ .

(True for  $1 \leq p \leq 10$ .)