Graph Theory Lecture 9 Nov. 9, 10

Date / /

Theorem 36 $ex(n; K_{s,t}) \le \frac{1}{2}(s-1)^{\frac{1}{t}} n^{2-\frac{1}{t}} + \frac{1}{2}(t-1)n$, i.e.,

$ex(n; K_{s,t}) \le \frac{1}{2} z(n, n; s, t)$.

Proof. (1st) Let G be an extremal graph such that $\|G\| = ex(n; K_{s,t})$.

Define a bipartite graph $H = (A, B)$ based on G. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$.

Let $A = \{a_1, a_2, \cdots, a_n\}$, $B = \{b_1, b_2, \cdots, b_n\}$ and $a_i \sim_H b_j$ if and only if $v_i \sim_G v_j$, see Figure 29 for an example. Now, clearly, $\|H\| = 2\|G\|$ and $a_i \not\sim_H b_i$ for $i = 1, 2, \cdots, n$. Moreover, if $G \not\supseteq K_{s,t}$, then $H \not\supseteq K_{s,t}$ (bipartite). (?) This concludes that

$\|G\| = \frac{1}{2}\|H\| \le \frac{1}{2} z(n, n; s, t)$. ■ 1st

(2nd proof) Two-way counting

Consider the number of stars $K_{1,t}$. Since $G \not\supseteq K_{s,t}$, every set of t vertices of $V(G)$ has at most $s-1$ centers of stars whose pendant vertices are these vertices. So, there are at most $(s-1)\binom{n}{t}$ t-stars. The number of stars $K_{1,t}$ can be obtained by $\sum_{i=1}^{n}\binom{d_i}{t}$ where $d_i = deg_G(v_i)$. Now, let $m = \|G\|$. $\sum_{i=1}^{n} d_i = 2m$. By a similar technique as Theorem 34, we conclude the proof (in Lecture 8) ■

平均度数 (*) $n \cdot \binom{2m/n}{t} \le \sum_{i=1}^{n}\binom{d_i}{t} \le (s-1)\binom{n}{t}$.

$\left(\binom{7}{3} + \binom{5}{3} \ge 2 \cdot \binom{6}{3}.\right)$

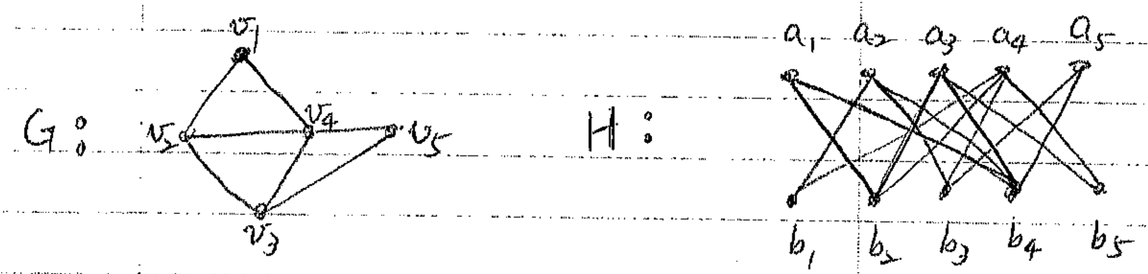


Figure 29. Bipartite version

Ramsey Theory

(∘) The Ramsey number $R(s,t)$ is the smallest value "n" for which either a graph G of order n contains K_s or $\overline{G} \geq K_t$.

(∘) Edge-coloring version of Ramsey number.

The Ramsey number $R(s,t)$ is the smallest value "n" for which in any 2-edge-colored K_n (red and blue), either there exists a red K_s or a blue K_t. (A red K_s is a complete graph of order s such that all its edges are colored red.)

(∘) $R(3,3)=6$ (Do you know this fact?)

Theorem 37 The following statements are true:

(1) $R(s,2)=s$ and $R(2,t)=t$,

如果等号成立，则不含K_s in G，及不含K_t in $\overline{G}$的图(极)点数为$R(s,t-1)+R(s-1,t)-1$。当$R(s,t-1)$及$R(s-1,t)$皆为偶数时不成立。(?)

(2) $R(s,t)=R(t,s)$,

(3) For $s>2$, $t>2$, $R(s,t)\le R(s,t-1)+R(s-1,t)$; and

(4) $R(s,t)\le\binom{s+t-2}{s-1}=\binom{s+t-2}{t-1}$.

and $R(s,t)\le R(s,t-1)+R(s-1,t)-1$ if both $R(s,t-1)$ and $R(s-1,t)$ are even

Proof. (1) and (2) are easy to see.

Claim of (3).

Let $n=R(s,t-1)+R(s-1,t)$. Then, in K_n, each vertex is of degree $R(s,t-1)+R(s-1,t)-1$. Therefore, if K_n is 2-edge-colored by red and blue, then the edges incident to a fixed vertex $x\in V(K_n)$ are either red edges or blue edges. By Pigeon-hole principle, either there are $R(s,t-1)$ blue edges or $R(s-1,t)$ red edges. If the first case holds, then in $\langle N_{K_n}(x)\rangle_{K_n}$ (a complete graph of order $R(s,t-1)$), either there exists a red K_s or blue K_{t-1}. Hence, we have a red K_s or a blue K_t in K_n. The other case can be obtained by a similar argument

Claim of (4) By inductive argument. (Or induction)

$R(s,t) \le R(s,t-1) + R(s-1,t)$

$$\le \binom{s+t-1-2}{s-1} + \binom{s-1+t-2}{t-1} = \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2}$$

$$= \binom{s+t-3+1}{s-1} = \binom{s+t-2}{s-1}.$$

$\blacksquare$

Theorem 38 (Erdős and Szekeres, 1935)

For each $s \ge 2$, $R(s) \le \frac{2^{2s-2}}{s^{1/2}}$. $(R(s) =_{def} R(s,s))$

Proof. $R(s,s) \le \binom{2s-2}{s-1}$. We claim $\binom{2s-2}{s-1} \le \frac{2^{2s-2}}{s^{1/2}}$ by induction on s. First, if $s=2$, $2 \le \frac{4}{\sqrt{2}}$, the assertion is true. Assume that the assertion is true for $s=k$, then $\binom{2k-2}{k-1} \le \frac{2^{2k-2}}{k^{1/2}}$.

Now, we calculate $\binom{2k}{k} = \frac{(2k)!}{k!\,k!} = \frac{2k\cdot(2k-1)\cdot(2k-2)!}{k^2\,(k-1)!\cdot(k-1)!}$

$$= \frac{2k(2k-1)}{k^2}\cdot\binom{2k-2}{k-1} \le \frac{4k^2-2k}{k^2}\cdot\frac{2^{2k-2}}{k^{1/2}} = \frac{(4k-2)}{4k}\cdot\frac{2^{2k}}{k^{1/2}}. \quad (1)$$

Since $(k+1)^{1/2} \le \frac{4k\cdot k^{1/2}}{4k-2}$, we conclude that $\binom{2k}{k} \le \frac{2^{2k}}{(k+1)^{1/2}}$.

$\blacksquare$

(•) The result has been there for almost 50 years before the improvement due to Thomason in 1988:

$R(s) \le 2^{2s}/s$.

(**) The original proof by Ramsey shows that

$$R(s) \le 2^{2s-3} = \frac{2^{2s-2}}{2}. \quad (1930)$$

<u>Theorem 39</u> $R(k) \ge \lceil 2^{\frac{k}{2}} \rceil$. $(k \ge 3)$

<u>Proof.</u> (Probabilistic method)

Consider a random red-blue coloring of the edges of K_n. For a fixed set T of k vertices, let A_T be the event that $\langle T \rangle_{K_n}$ is monochromatic. Hence $P(A_T) = (\frac{1}{2})^{\binom{k}{2}} \cdot \underset{\substack{\uparrow \\ \text{red or blue}}}{2} = 2^{1-\binom{k}{2}}$.

Since there are $\binom{n}{k}$ possible sets for T, the probability that at least one of A_T occurs is $\underset{(\text{期望值})}{\binom{n}{k} \cdot 2^{1-\binom{k}{2}}}$. Now, if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then no event A_T occurs is of positive probability, i.e., there exists a coloring of edges such that no monochromatic K_k occurs. Therefore, for such n, $R(k) > n$.

Let $n = \lfloor 2^{\frac{k}{2}} \rfloor$. (It suffices to show that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.)

$$\binom{n}{k} 2^{1-\binom{k}{2}} < \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \qquad \left(1-\binom{k}{2} = 1 - \frac{k^2}{2} + \frac{k}{2}\right)$$

$$\le \frac{(2^{\frac{k}{2}})^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \le \frac{2^{1+\frac{k}{2}}}{k!} < 1 \quad (k \ge 3).$$

Hence, $R(k) \ge \lceil 2^{\frac{k}{2}} \rceil$. ∎

(∘) Combining Theorems obtained above

$$2^{\frac{s}{2}} \leq R(s) \leq 2^{2s-3} \quad \text{for } s \geq 2.$$

(∘∘) Open problem: $R(s) = 2^{(c+o(1))s}$, (c may be equal to 1).

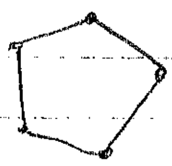
Theorem 40 Known results of $R(s,t)$.

$t \backslash s$	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	36–41	49–61	59–84	73–115
5			43–48	58–87	80–143	101–216	133–316
6				102–165	115–298	134–495	183–780
7					205–540	217–1031	252–1713
8						282–1870	329–3583
9							565–6588

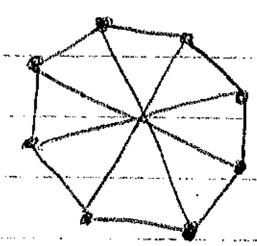
$R(s,t) = R(t,s)$

(∘) The results of lower bounds are obtained by "a special edge-coloring" with two colors. Corresponding to the coloring we have G and $\overline{G}$ of order (prescribed).

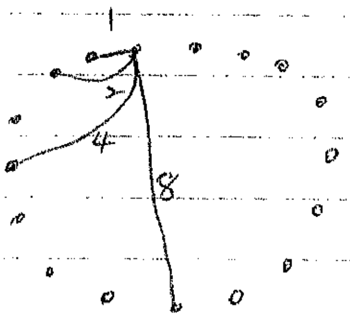
$R(3,3)$: (>5)



$R(3,4)$: (>8)



$R(4,4)$ (>17)



$G(17; \{1,2,4,8\})$

Problem Find as many vertices (n) as possible such that a graph G of order n satisfying $G \not\supseteq K_5$ and $\overline{G} \not\supseteq K_5$.

(Try 43 !)

Exercise 2-6 Find a better lower bound and a better upper bound for $R(5)$.

(*) Ramsey number : Not limited to graphs ! There are versions on Algebra, Geometry and Number Theory.

Theorem 4.1 $R(p_1, p_2, \cdots, p_t) \le R(p_1 - 1, p_2, \cdots, p_t) + R(p_1, p_2 - 1, p_3, \cdots, p_t) +$

$\cdots + R(p_1, p_2, \cdots, p_t - 1) - t + 2$.

Proof. By a similar argument as the proof $R(s, t) \le R(s-1, t) + R(s, t-1) - 1$. ∎

(·) $R(3,3,3) \le 6+6+6-3+2 = 17$ (Theorem 4.1).

(··) There exists a 3-edge-coloring of K_{16} such that no mono-chromatic triangles occur.

Theorem 4.2 $\underbrace{R(3,3,\cdots,3)}_{k\text{-tuples}} \stackrel{def}{=} R_k(3) \le \lfloor e \cdot k! \rfloor + 1$.

Proof. Since $R(3,3) = 6$, $R(3,3,3) = 17$, the assertion is true for $k = 2$ and 3. Assume that it holds for $k-1$ when $k > 3$. Hence, $R_{k-1}(3) \le \lfloor e(k-1)! \rfloor + 1$. By Theorem 4.1,

$$R_k(3) \le k(\lfloor e(k-1)! \rfloor + 1) - k + 2$$

$$= k\lfloor e(k-1)! \rfloor + 2.$$

Now, $k\lfloor e(k-1)! \rfloor = k\lfloor (k-1)! \cdot (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(k-1)!} + \frac{1}{k!} + \cdots) \rfloor =$

$= k\lfloor M + \frac{1}{k} + \frac{1}{(k+1)k} + \frac{1}{(k+2)\cdot(k+1)k} + \cdots \rfloor = \cdots$

$(?) \doteqdot \lfloor (k!) \cdot e \rfloor - 1$. ∎

(•) Instead of $R(s,t)$, we use $R(H_1,H_2)$ to denote the smallest integer n such that any 2-edge-coloring (red, blue) of K_n, either there exists a red H_1 or a blue H_2.

<u>Theorem 43</u> $R(C_4,C_4) =_{\text{def}} R(C_4) \le 8$.

<u>Proof.</u> Consider a graph $G \not\supseteq C_4$ and $|G| = 8$. By Theorem 35-1 $\|G\| \le \frac{1}{2} z(8,8;2,2) \le \frac{n\cdot(1+\sqrt{4n-3})}{4} = 2(1+\sqrt{29}) < 14$. That is, if a graph G is of order 8 and size 14, then $G \supseteq C_4$. Now, in a 2-edge-colored K_8, either red or blue edges induce such a graph, the proof follows. ∎

<u>Theorem 44</u> $R_k(C_4) \le k^2+k+2$.

<u>Proof.</u> Let $n = k^2+k+2$ and consider a k-edge-colored K_n. Since $ex(n;C_4) \le \frac{n}{4}(1+\sqrt{4n-3}) = \frac{k^2+k+2}{4}\cdot(1+\sqrt{4k^2+4k+5})$

$\le \frac{k^2+k+2}{2} \cdot \frac{1+\sqrt{4k^2+4k+5}}{2}$.

Now, compare (k^2+k+1) and $\frac{k(1+\sqrt{4k^2+4k+5})}{2}$. By direct calculation we have $k^2+k+1 > \frac{k(1+\sqrt{4k^2+4k+5})}{2}$. This implies that

$k\cdot ex(n;C_4) \le \frac{k^2+k+2}{2} \cdot \frac{k(1+\sqrt{4k^2+4k+5})}{2} < \frac{(k^2+k+2)(k^2+k+1)}{2} = \binom{n}{2}$. ∎

Date (•) $R(H_1, H_2) = \min\{n \mid |G| = n,\ G \geq H_1 \text{ or } \overline{G} \geq H_2\}$.

Theorem 45

For $l \geq 1$ and $p \geq 2$, $R(lK_2, K_p) = 2l + p - 2$.

Proof. Let O_k denote the graph of order k which contains no edges.
(Stable set)
Let H be a graph of order $2l+p-3$ such that $H = O_{p-2} \cup K_{2l-1}$ (see Figure 30).

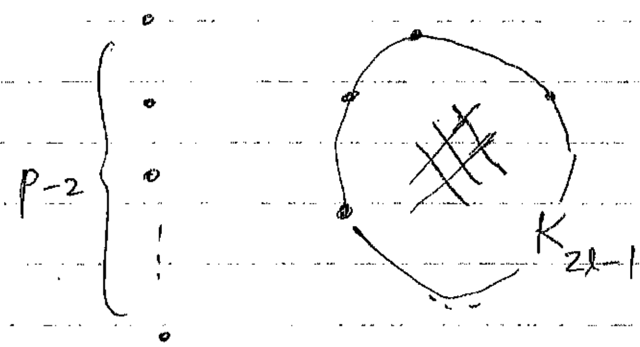


Figure 30

So, it is not difficult to see that H does not contain a matching of size l. As a consequence, $\overline{H} = O_{2l-1} \vee K_{p-2}$ (join) does not contain a K_p. Hence, $R(lK_2, K_p) \geq 2l + p - 2$.

On the other direction of inequality, let $n = 2l + p - 2$ and assume that a graph G of order n does not contain a matching of size l. Let $s < l$ be the maximum number of independent edges in G.

Now, consider $\overline{G}$. Since in G, the set of vertices not incident to these s edges induces a graph of order $n-s$ which has no edges, $\overline{G}$ contains a complete graph of order $n-2s \geq n-2(l-1) = p$. This concludes the proof. ■

Appendix (For your reference in writing exercises!)

Use max-flow min-cut theorem to prove Hall's Theorem.

Proof. Let $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, $m \le n$.

Claim: There exists a matching in $G = (X, Y)$ saturates X if and only if for each subset $A \subseteq X$, $|\Gamma(A)| \ge |A|$.

(Note) We shall prove the theorem for both $(\Rightarrow)$ and $(\Leftarrow)$ by using network argument.

First, we construct a network N by (1) adding s and t such that (s, x_i) and (y_j, t) are arcs in N, $1 \le i \le m$ and $1 \le j \le n$; (2) orienting $x_i y_j \in E(G)$ with (x_i, y_j), and (3) $c(s, x_i) = c(y_j, t) = 1$ and $c(x_i, y_j) = M > |X| = m$. (See Figure 1 below.)

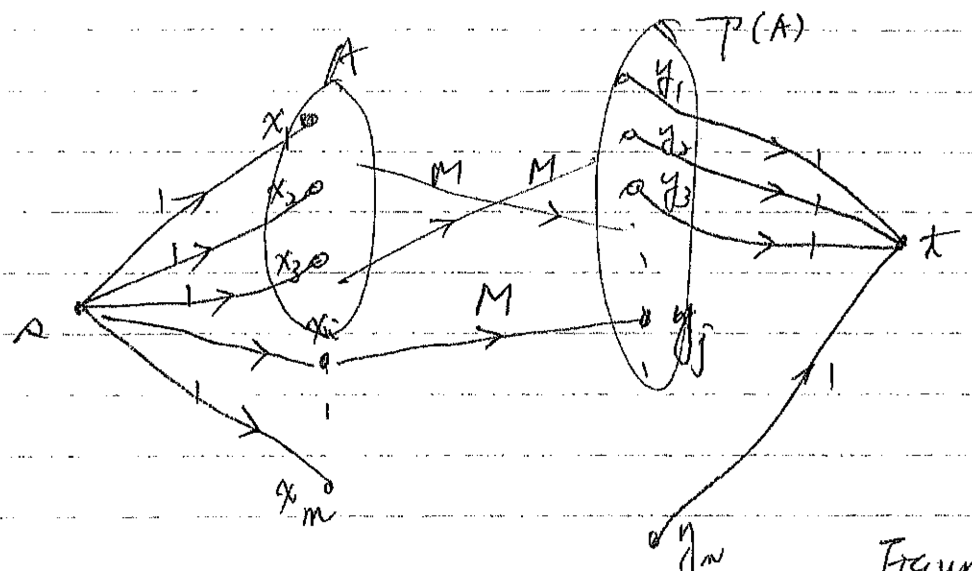


Figure 1.

Now, it is easy to see that a flow of value $|X| = m$ will provide a matching saturates X since each y_j can take at most one flow value through the flow.

$(\Rightarrow)$ Suppose that there exists a subset A of X such that $|A| > |\Gamma(A)|$. Let $S = \{s\} \cup A \cup \Gamma(A)$. Then, $\langle S, \bar{S} \rangle$ is a cut with capacity $\underbrace{(|X| - |A|)}_{(s \to \bar{S})} + \underbrace{|\Gamma(A)|}_{(S \to t)}$ since there are no arcs from the vertices of A to the vertices of $Y \setminus \Gamma(A)$. This capacity of cut is less than $|X|$, Hence, there exists no flow with value $|X|$ and thus no matching saturates X.

$(\Leftarrow)$ Assume that $|\Gamma(A)| \geq |A|$ for each $A \subseteq X$. It suffices to claim that all cuts have capacity at least $|X|$. Let $S = \{s\} \cup A \cup B$, see Figure 2. ($\underline{A} \subseteq X$ and $\underline{B} \subseteq Y$.) ← They can be empty.

(B is not necessarily be $\Gamma(A)$.)

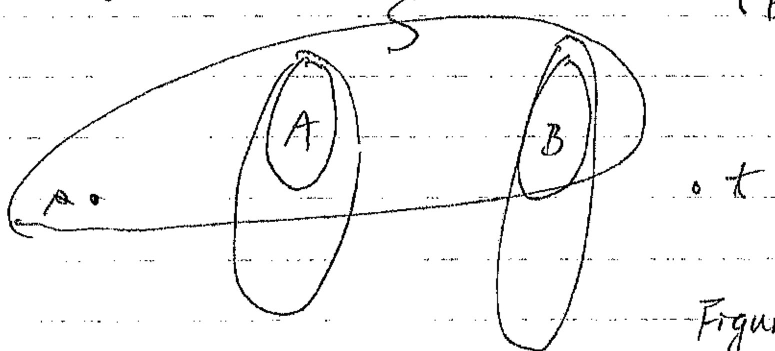


Figure 2.

Now, if there exists an arc from the vertices of A into $Y \setminus B$, then $c(S, \bar{S}) \geq M > |X|$. On the other hand, if $\Gamma(A) \subseteq B$, then $c(S,\bar{S}) = (\underbrace{|X| - |A|}_{s \to \bar{S}}) + |B| \geq |X| - |A| + |\Gamma(A)| \geq |X|$. Since $c(S,\bar{S}) = |X|$ in the case $S = \{s\}$, we obtain a min-cut with capacity $|X|$ and thus there exists a flow with maximum value $|X|$. The proof follows. ∎

Use max-flow min-cut theorem to prove Menger's Theorem.

Proof. First, we prove a directed version of Menger's Theorem.

(i) If s and t are distinct vertices of a digraph D such that $s \not\to_D t$, then the maximum number of internally disjoint u-v directed paths in D equals the minimum number of vertices in a u-v separating set in D.

(ii) For the undirected version, we replace each edge uv by a pair of arcs (u,v) and (v,u).

Proof of (i).

Let $\tilde{D}$ be the digraph obtained as follows:

(1) $\forall\ x \in V(D) \setminus \{s,t\}$, split x into two vertices x' and x'', also let $(x', x'') \in A(\tilde{D})$;

(2) $\forall\ (x,y) \in A(D)$ such that $\{x,y\} \cap \{s,t\} = \phi$, replace (x,y) with (x'', y');

(3) For $x \neq t$, $(s,x) \in A(D)$, $(x,s) \in A(D)$, replace them with (s,x') and (x'',s) respectively, and

(4) Replace (t, x) with (t, x') and (x, t) with (x'', t) if $x \neq s$.

As a consequence, we have a network N defined on D by assigning each arc a capacity "1".

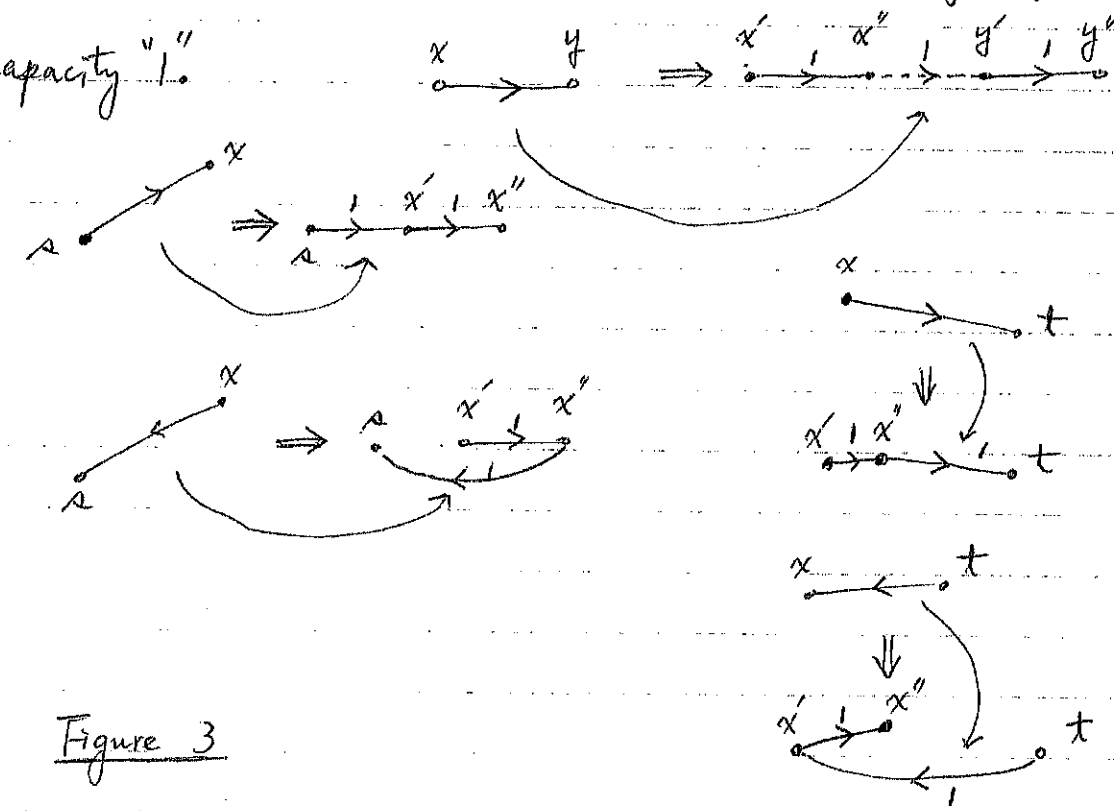


Figure 3

Let m be the maximum number of internally disjoint $s-t$ paths in D and n be the minimum number of vertices in an $s-t$ separating set in D. Let A be a $u-v$ separating set of arcs in $\bar{D}$ and $|A| = n'$.

First, we observe that if $(S, \bar{S})$ is a cut, then $cap(S, \bar{S}) \leq n'$. Here, S contains s if $(s, x') \in A$, S contains x if $(x', x'') \in A$ and $(x'', t) \in A$.

Moreover, $cap(S, \bar{S}) \geq n$. As a matter of fact, the min-cut is of capacity n and thus, N has a maximum flow n. By the way, the construction of network shows that a flow value 1 will give a (directed) path from s to t. Since each arc has capacity one, those paths are internally disjoint. The proof follows. ∎