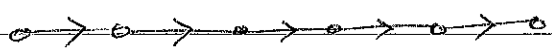
Review

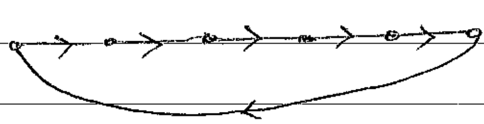
(•) A directed graph D (digraph) is an ordered pair (V, A) where V is the set of vertices and A is a set of ordered pairs (u,v) where $u, v \in V$, (u,v) is called an arc, denoted also by $\overrightarrow{uv}$.

(•) Clearly, (u,v) is different from (v,u).

Definition (Connectedness)

A digraph graph $D = (V, A)$ is strongly connected if for any two vertices u and v, there exists a directed path (dipath) from u to v. D is weakly connected if for any two vertices u and v, there exists either a dipath from u to v or a dipath from v to u.

(•)  weakly connected

 strongly connected.

Networks play an important role in applications.

Definition (Network)

A network N is a digraph D with two distinguished vertices s and t, called the <u>source</u> and <u>sink</u> of N, respectively, and a non-negative integer-valued function c on $E(D)$. The digraph is the underlying digraph of N and the function c is the capacity function on N. For convenience, $c(\vec{a}) = c((x,y)) = c(x,y)$ for each arc $\vec{a} = (x,y)$, is the capacity of $\vec{a}$.

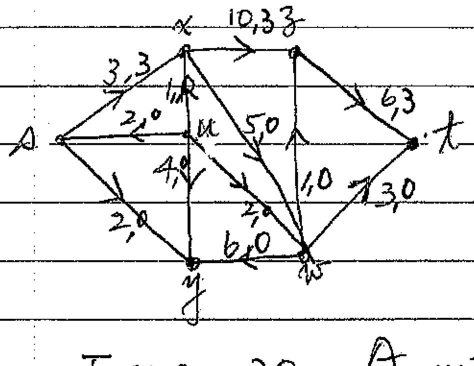


Figure 20. A network with capacity

$N^+(x) = \{y \mid y \in V(D) \text{ and } (x,y) \in A(D)\}$, $\deg^+(x) = |N^+(x)|$ (outdegree of x)

$N^-(x) = \{y \mid y \in V(D) \text{ and } (y,x) \in A(D)\}$, $\deg^-(x) = |N^-(x)|$ (indegree of x)

(•) A flow in a network N, f, is a function on $A(D)$, s.t.

$(f: A(D) \longrightarrow \mathbb{R}^+ \cup \{0\})$

① $0 \le f(\vec{a}) \le c(\vec{a})$ for every $\vec{a} \in E(D)$, and
(capacity bound)

② $\sum_{y \in N^+(x)} f(x,y) = \sum_{y \in N^-(x)} f(y,x)$ for every $x \in V(D) \setminus \{s,t\}$.
(Conservation law)

- The net flow into x is equal to $\sum_{y \in N^-(x)} f(y,x) - \sum_{y \in N^+(x)} f(x,y)$ which is zero except $x \in \{s,t\}$.

- $(V_1, V_2) = \{(x,y) \in E(D) \mid x \in V_1 \text{ and } y \in V_2\}$ (digraph version!)
  ↳ (Only arcs from V_1 to V_2)

(••) A cut in N is $(X, \underbrace{V(D) \setminus X}_{= X'})$ such that $s \in X$ and $t \in V(D) \setminus X$.

Definition Let $K = (X, X')$ be a cut in N. Then, the capacity of K, $\operatorname{cap} K = c(X, X') = \sum_{(x,y) \in K} c(x,y)$.

e.g. In Figure 20, let $X = \{s, x, u, y\}$, then $c(X, X') = 17$.

(*) Definition The value of a flow f in N is defined as the net value flow out the source and therefore the net value flow into the sink. (Denoted by $\operatorname{val} f$)

Theorem 21 Let f be a flow in a network N and $K = (X, X')$ be a cut in N. Then, $\operatorname{val} f \le \operatorname{cap} K$.

Proof. Note that $\mathrm{val}\, f = f(s, \{s\}') - f(\{s\}', s)$ and

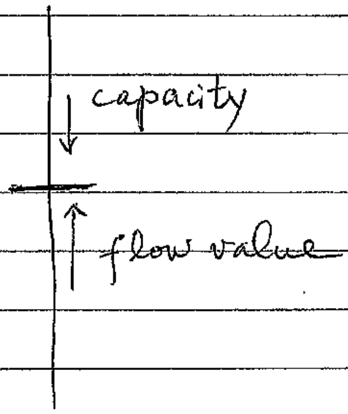
$$f(x, \{x\}') - f(\{x\}', x) = 0, \quad \forall x \in X \setminus \{s\}. \quad (s \in X,\ t \in X')$$

This implies that $\underbrace{\sum_{x \in X} \left[f(x, \{x\}') - f(\{x\}', x) \right]}_{= f(X, X') - f(X', X) \ \text{(Calculation)}} = \mathrm{val}\, f.$

Now, $f(X, X') \le \mathrm{cap}(X, X')$ and $f(X', X) \ge 0$. Hence,

$\mathrm{val}\, f \le \mathrm{cap}\, K$.

- A minimum cut is a cut K in N such that for every cut K' in N, $\mathrm{cap}\, K \le \mathrm{cap}\, K'$.
- A maximum flow is a flow f in N such that for every flow f' in N, $\mathrm{val}\, f \ge \mathrm{val}\, f'$.



$\Rightarrow$ If there exist a K and an f s.t. $\mathrm{cap}\, K = \mathrm{val}\, f$, then K is a minimum cut and f is a maximum flow.

min-max problem

Theorem 22 (Ford and Fulkson, 1956–1962)

In any network N defined on D with source s and sink t the value of a maximum flow equals the the capacity of a minimum cut.

Proof. Clearly, if there exist no cuts K such that its capacity of the cut is (at least) val f, then f does not exist. (Theorem 21) val $f \le$ cap K So, it suffices to claim that if the value of a maximum flow f is v, then there exists a cut K, such that cap K = val $f = v$.

(*) Let f be a maximum flow of N.

→ Define a subset $S \subseteq V(D)$ recursively as follows. Let $s \in S$.

If $x \in S$, and $c(x,y) > f(x,y)$ or $f(y,x) > 0$, then let $y \in S$.

We shall prove that (S, S') is a cut with capacity v.

First, we claim $t \notin S$. Suppose not, i.e., $t \in S$. Hence, we can find a sequence of vertices in N such that $s = x_0, x_1, \cdots, x_l = t$.

Moreover, if we let $\varepsilon_i = \max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\}$, $i = 0, 1, \cdots, l-1$, then $\varepsilon_i > 0$. Let $\varepsilon = \min_i \{\varepsilon_i\}$. Now, let

$f^*(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \varepsilon$ if $c(x_i, x_{i+1}) - f(x_i, x_{i+1}) = \varepsilon_i > 0$ and

$f^*(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \varepsilon$ if $f(x_{i+1}, x_i) = \varepsilon_i > 0$. As a consequence, f^* is a flow from s into t with value $\operatorname{val} f^* = v + \varepsilon$, a contradiction. (See Figure 21) By the definition of a flow,

$$\operatorname{val} f = v = \sum_{x \in S, y \in S'} f(x,y) - \sum_{x \in S', y \in S} f(x,y). \qquad (1)$$

Again, by the definition of S, if $x \in S$ and $y \in S'$, then $c(x,y) = f(x,y)$ and $f(y,x) = 0$. This implies that $(1) = \sum_{x \in S, y \in S'} c(x,y)$, the proof follows. $v = cap(S, S')$ ∎

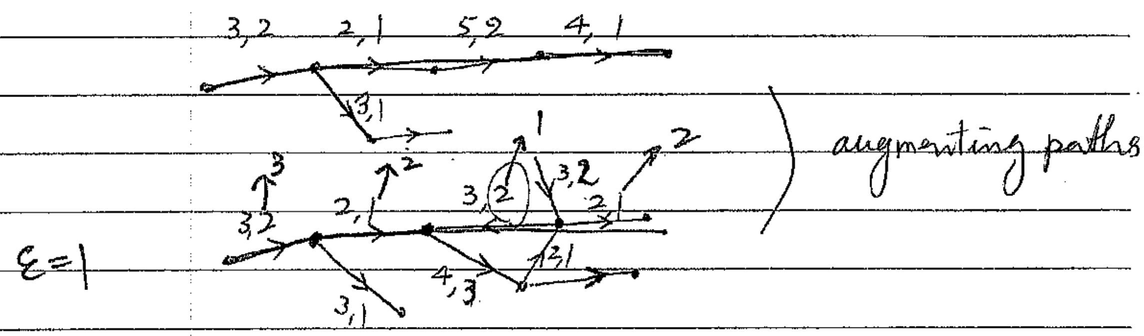


Figure 21 augmenting flow f^*

Theorem 23 (Menger, 1927)

Let s and t be two nonadjacent vertices of a graph G. Then, the minimal number of vertices separating s from t is equal to the maximal number of vertex-disjoint $s-t$ paths. (Internally)

Proof. (1st)

Let the number of vertices separating s and t be k. Then, it is easy to see that there are ① at most k independent paths connecting (vertex-disjoint) s and t. Also, if $k=1$, then we have a path joining s and t. Now, suppose the assertion is not true for $k \geq 2$, i.e., we can find less than k independent s-t paths for certain k. Now, take the minimal k (≥ 2) in which we have a counterexample. Then among all such examples let G be the one with minimum size (number of edges).

First, we notice that ② s and t have at most $k-1$ independent paths and no common neighbor. For otherwise, let sx and xt be edges of G. Then, $G-x$ will be a counterexample for "$k-1$" (smaller than k).

Let W be a separating set of s and t and $|W| = k$. Suppose, neither $N_G(s) = W$ nor $N_G(t) = W$. (Figure 22-1)

G:

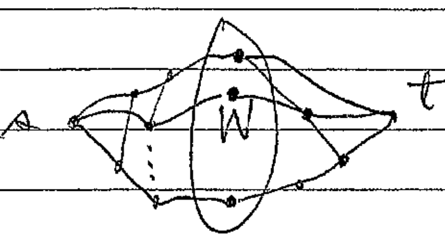


Figure 22-1.

Let G_s be obtained by deleting all the vertices to the left of G in Figure 22-1 and adding a replacing s' with edges joining to W, see Figure 22-2. Now, G_s has fewer edges than G and thus there are k independent s'-t paths. Hence, we have k W-t independent paths. With the same technique, we derive
(新圖不再是反例)
k s-W independent paths (by changing s to t).

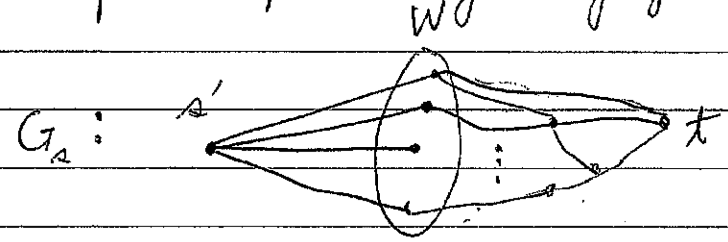


Figure 22-2.

③ So, as a conclusion, either s or t must have their neighbors W. Let $N_G(s) = W$ and $P = \langle s, x_1, x_2, \cdots, x_l, t \rangle$ be a shortest s-t path. Then, $l \geq 2$. Consider $G - x_1 x_2$.

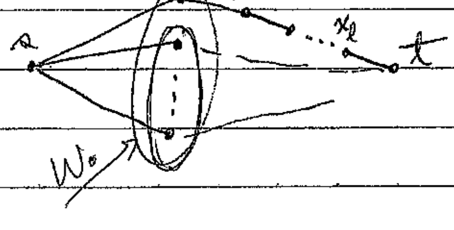


Figure 22-3.

In $G - x_1x_2$, there exists an s-t separating set W_0 of size $k-1$. Then, both $W_1 = W_0 \cup \{x_1\}$ and $W_2 = W_0 \cup \{x_2\}$ are s-t separating sets of G. (4)

By the fact that P is a shortest s-t path, s is not adjacent to x_2 and t is not adjacent to x_1. This implies that $N_G(s) = W_1$ since t is not adjacent to a vertex of the separating set W_1.

Similarly, $N_G(t) = W_2$. Hence, $N_G(s) \cap N_G(t) = W_0$, (5) a contradiction.

($|W_0| = k-1 \geq 1$) (s and t have common neighbors.) $|W_0| = k-1 > 0$ ∎

(2nd proof.) Ex. 2-1. (請參考其它圖論課本。)

(•) The "Edge" version of Menger's theorem can be stated as follows:

Let s and t be two vertices of G. Then, the minimal number of edges separating s from t is equal to the maximal number of edge-disjoint s-t paths. (We can prove this part by using Theorem 22. Replace each edge xy of G by (x,y) and (y,x) and assign capacity "1" to each arcs.)

Theorem 24 (Set version of Menger's Theorem)

If S and T are abitrary subsets of $V(G)$, then the maximal number of vertex-disjoint (including endvertices) $S-T$ paths is $\min\{|W| \mid W \subseteq V(G)$ and $G-W$ has no $S-T$ paths$\}$.

Proof. By adding to new vertices s and t as in Figure 23, we have a new graph $\tilde{G}$. Now, by Menger's Theorem, the maximal number of $S-T$ paths is the same as that of $s-t$ paths in $\tilde{G}$. Hence, we have the proof. ∎

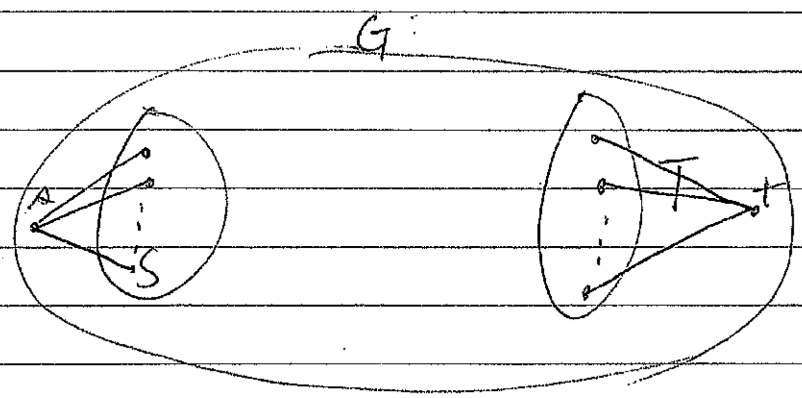


Figure 23, graph $\tilde{G}$