

- A maximal connected subgraph without a cut vertex is called a "block".
 \downarrow
 increase the # of components

(**) Every block of a graph G is either a maximal 2-connected subgraph or a bridge (with its endvertices), or an isolated vertex.
 (其定義的定義可能不同)
 (See Figure 16.)

- A block graph of G is a bipartite graph (A, B) where A is the set of cutvertices of G and B is the set of blocks, and $a \in A$ is incident to $B_i \in B$ if $a \in V(B_i)$.
 $bc(G) =$

Theorem 16 The block graph of a ^{connected} graph is a tree.

Proof. Observe that each block is an induced subgraph and any two blocks have at most one cutvertex in common.

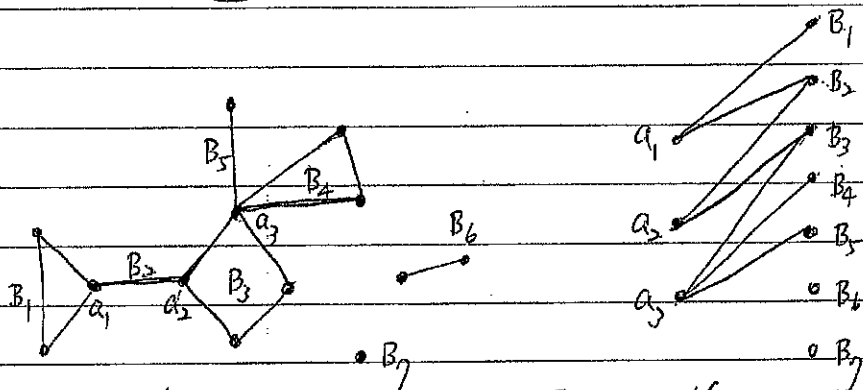
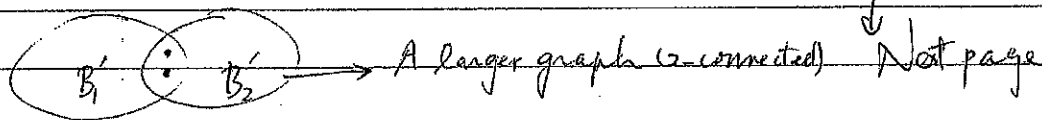


Figure 16. A block graph

Since G is connected, if G has only one block and thus contains no cutvertex, $bc(G)$ is a single vertex. The proof is trivial. Assume

that G contains more than one block. Then, each vertex is either in a block or a cutvertex itself. Now, consider a cutvertex v .

and a block B_i . Let $u \in B_i \setminus \{v\}$. ($B_i \setminus \{v\}$ is non-empty since G is connected.) Then, we have a path P connecting v and u . Clearly, v is going to connect to a block which contains some vertices of P .

If this block B_i , then vB_i is an edge of $bc(G)$, done. Otherwise, this path will contain a cutvertex following the block and travels to another block and finally to B_i . The other two cases, cutvertex to cutvertex and block to block can be verified similarly.

Now, for the acyclic part, a cycle in $bc(G)$ will produce a cycle in G which passes all cutvertices involved. But, in that case, none of these cutvertices are cutvertices anymore, a contradiction.

This completes the proof. ■

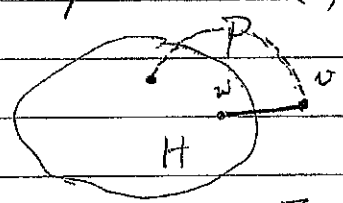
Theorem 17 A graph G is 2-connected if and only if it can be constructed from a cycle by successively adding H -paths to graphs H already constructed.

Proof. (\Leftarrow) From the construction, it is clear that G contains no cutvertices. Hence, G is 2-connected.

(\Rightarrow) Assume that G is 2-connected and H is a maximum (size) subgraph following the construction. This is possible, since G contains a cycle. In fact, H is an induced subgraph, since for any two vertices x and y in $V(H)$ and $xy \in E(G) \setminus E(H)$, we have an H -path and a larger subgraph will be obtained.

Now, assume that $H \neq G$. $\exists v \in V(G) \setminus V(H)$; $w \in V(H)$ and $vw \in E(G) \setminus E(H)$. Since G is 2-connected, $G - w$ contains a

v - H path P . (See Figure 17) This implies that $\langle vw, P \rangle$ is an H -path. Hence, a larger



subgraph is obtained, a contradiction. \blacksquare

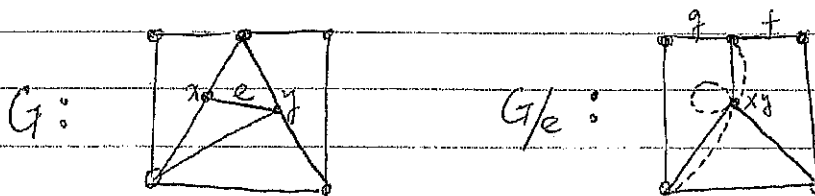
Figure 17.
An extra H -path

Definition (Graph minors)

A graph M is called a minor of G if M can be obtained from G by contracting edges, deleting vertices and edges.

Review (Edge-contraction)

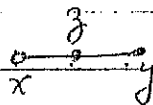
Given an edge $xy \stackrel{=}{=} e$ of a graph G , the graph G/e is obtained from G by contracting e ; that is to identify the vertices x and y and ^{deleting} resulting loops and duplicate edges.



Example K_4 is a minor of the above G .
(Contracting e, f and g .)

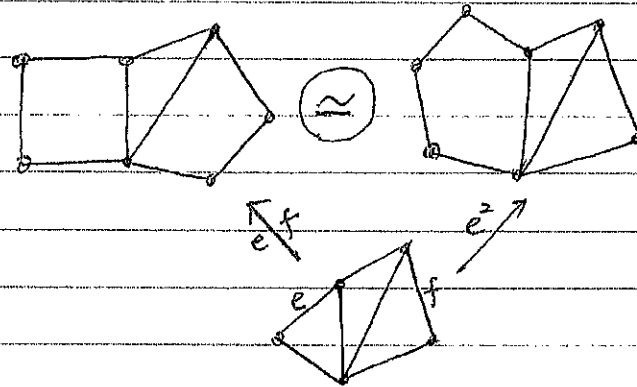
Definition (Subdivision)

A subdivision of an edge xy is obtained by adding a new vertex z such that we have edges xz and zy .



Definition (Homeomorphic Graphs)

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

Remark: Two cycles are homeomorphic.

(*) The reverse of subdivision can be "considered" as contraction.



Theorem 18 If G is 3-connected and $|G| > 4$, then G has an edge such that G/e is again 3-connected.

Proof. Suppose not. Then, for each $xy \in E(G)$, the graph G/xy contains a separator S with $|S| \leq 2$. ($G-S$ is disconnected.)

Since $\kappa(G) \geq 3$, the contracted vertex $v_{xy} \in S$ and $|S| = 2$, i.e.,

$\exists z \in V(G)$, s.t. $S = \{v_{xy}, z\}$. Therefore, $T = \{x, y, z\}$ is a separator set

of G . Since no proper subset of T can separate G , each vertex

of T is incident to every component of $G-S$. (See, Figure 18.)

Among all edges of G , we choose an edge xy and its corresponding vertex z such that the component C has minimum size. (*)

Let $v \in V(C)$ and $vz \in E(G)$. By assumption, G/vz is again not 3-connected and there exists a corresponding vertex w such that

$\{v, z, w\}$ separates G .

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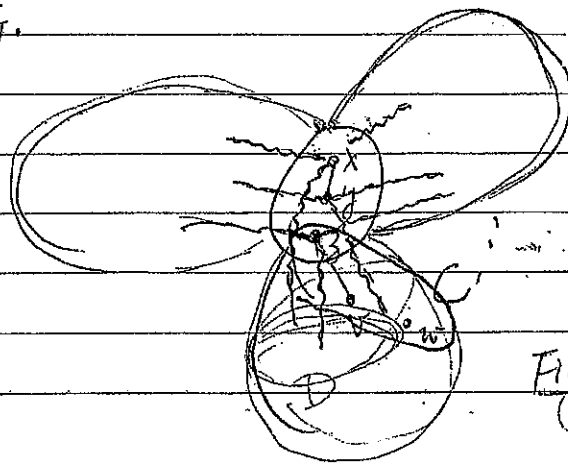


Figure 18 3-connected graph

Moreover, each vertex of $\{v, z, w\}$ is incident to every component of $G - \{v, z, w\}$. Since $xy \in E(G)$, $G - \{v, z, w\}$ has a component D s.t. $D \cap \{x, y\} = \emptyset$. By the fact $v \in V(C)$, the neighbor of v in D is also in C . Hence, $D \cap C \neq \emptyset$. This implies that D is a proper subset of C , i.e., $|D| < |C|$, a contradiction to the choice of C . ($|C|$ is minimum.) \blacksquare

Theorem 19 (Tutte, 1961)

A graph G is 3-connected if and only if there exists a sequence G_0, G_1, \dots, G_n of graphs satisfying:

- $G_0 = K_4$ and $G_n = G$; and ^{of x, y}
(with degree _{at least 3})
- G_{i+1} has an edge xy such that $G_i = G_{i+1}/xy$, $1 \leq i < n$.

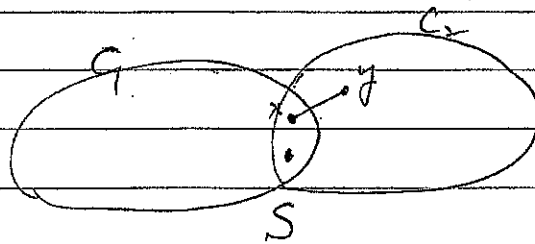
Proof (\Rightarrow) By Theorem 18, we start with G as G_n and end at $K_4 = G_0$.

(\Leftarrow) Let G_0, G_1, \dots, G_n be a sequence of graphs satisfying (a) and (b).

It suffices to show that if $G_i = G_{i+1}/xy$ is 3-connected, then

G_{i+1} is also 3-connected, for all $1 \leq i < n$.

Suppose not. Let S be a separator with $|S| \leq 2$. Also, let C_1 and C_2 be two components of $G_{i+1} - S$. Since $xy \in E(G_{i+1})$, let $\{x, y\} \cap V(C_1) = \emptyset$. (Figure 19) Now, if $\{x, y\} \subseteq C_2$, then $G_i - S$ is disconnected, a contradiction. Hence, at most one of $\{x, y\}$ is in C_2 , either x or y , but not both. Furthermore, if $v \notin \{x, y\}$ and $v \in V(C_1)$, then $G_i - S$ is also disconnected, a contradiction to the fact G_i is 3-connected. Hence, C_2 contains exactly one vertex of degree at most

2, $\rightarrow \leftarrow$.Figure 19, G_{i+1}

(Review)

Theorem 20 Every non-trivial graph G contains at least two vertices

which are not cutvertices.

Proof. Review that if v is a cutvertex of G , then the number of components of G , $c(G)$ is smaller than that of $c(G - v)$.

Now, consider u and v such that $d(u, v) = \text{diam}(G)$. We show both u and v are not cutvertices. Suppose not. Let u be a cutvertex, then $G - u$ is disconnected. Let w be a vertex which is in a component different from v belongs. Since u is a cutvertex and v, w are in different components, all $v-w$ paths must pass through u . This implies that $d(v, w) > d(v, u) = \text{diam}(G)$. Hence, u can not be a cutvertex. Similarly, v is not a cutvertex either. ■

(*) Determine whether a graph G contains a minor H is considered as the most important problem in the study of graph structure.