Theorem 11 $(V(G)=\{v_1, v_2, \cdots, v_n\})$

The vertices of a connected graph G can be enumerated so that $G_i =_{def} \langle\{v_1, v_2, \cdots, v_i\}\rangle_G$ is connected for every $i=1,2,\cdots,n$.

Proof. We construct all G_i's recursively. Define $\langle\{v_1\}\rangle_G$ be G_1 where v_1 is an arbitrary vertex of G. Assume that $G_1, G_2, \cdots, G_i$ have been constructed for $i \geq 1$ and $i<n$. Now, let $v \in V(G) \setminus V(G_i)$. Since G is connected, there exists a path connecting v and v_1. Let v_{i+1} be the last vertex on the path not in G_i. (See Figure 13) Then, $\{v_1, v_2, \cdots, v_i, v_{i+1}\}$ induces a connected graph, G_{i+1}. ▨

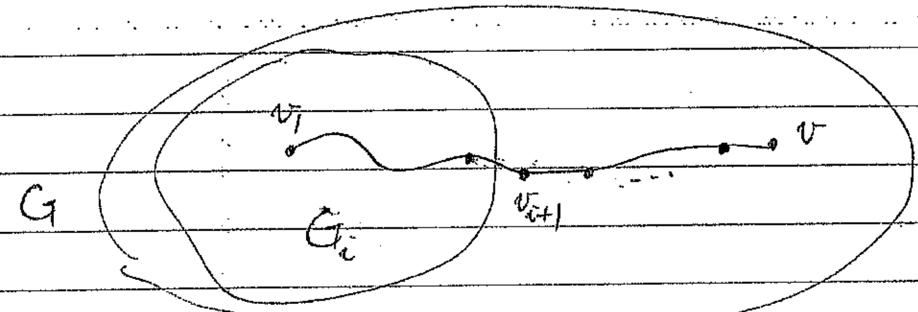


Figure 13. Graph enumeration

(•) The vertices of a tree T can be enumerated such that every v_{i+1} has a unique neighbor in $\langle\{v_1, v_2, \cdots, v_i\}\rangle_T$. ($v \sim v_i$ is unique.)

Theorem 12

If T is a tree and G is any graph with $\delta(G) \geq \|T\|$, then $T \leq G$.

(1st)
Proof. By induction on $\|T\|$. Clearly, the assertion is true for $\|T\|=1$. Assume that the assertion is true for $\|T\|=k \geq 2$. Consider a tree T with $\|T\|=k+1$. Since $\delta(G) \geq k+1 > k$, G contain a tree (arbitrarily given) of size k, T'. (Let $T' = T - v_0$ where v_0 is a pendant vertex of T, moreover $v_0 v' \in E(T)$.) Now, consider T' in G. v' corresponds to a vertex u in $V(G)$. Since $\delta(G) \geq k+1$, u has a neighbor u_0 which is not a vertex of T'. Note here that T' has k vertices not including u. By attaching u_0 to T', we have a subgraph T. ▨

Direct construction

(2nd) Proof. Let T be a tree defined on $\{v_1, v_2, \cdots, v_n\}$ such that for each i, every vertex v_{i+1} has a unique neighbor in $\langle\{v_1, v_2, \cdots, v_i\}\rangle_T$.
Now, we can construct directly on G the tree T by following the enumerating order. (Pick the vertices and edges recursively.)

Since $\delta(G) \geq \|T\|$, there is always a vertex in G available for the next choice. ∎

Theorem 13 $\qquad$ (∘) $\varepsilon(G) = \frac{\|G\|}{|G|}$, average size of a vertex.

Every graph G with at least one edge has an (induced) subgraph H with $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.

Proof. We construct a sequence $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$ of induced subgraphs of G as follows. If G_i has a vertex v_i of degree $\deg_{G_i}(v_i) \leq \varepsilon(G_i)$, let $G_{i+1} :=_{def} G_i - v_i$; if not, we terminate our sequence and set $H =_{def} G_i$.

Now, by the selection process, $\delta(H) > \varepsilon(H)$. Furthermore,

$$\varepsilon(G_{i+1}) \geq \frac{\|G_{i+1}\|}{|G_i|-1} = \frac{\|G_i\| - \deg_{G_i}(v_i)}{|G_i|-1} \geq \frac{\|G_i\| - \varepsilon(G_i)}{|G_i|-1}$$

$$= \frac{\|G_i\| - \|G_i\|/|G_i|}{|G_i|-1} = \frac{\|G_i\|\left(1-\frac{1}{|G_i|}\right)}{|G_i|-1} = \frac{\|G_i\|}{|G_i|} = \varepsilon(G_i).$$

Hence, $\varepsilon(H) \geq \varepsilon(G)$. ∎

(∘) If there exists no v_i to delete, then G is the subgraph we need.

(∗) $\qquad G' - v$ is an induced subgraph of G' where $v \in V(G')$.

一般而言，確定一个圖具有何种 Induced Subgraph 並不容易。

以下是目前尚未完全解决的問题，在 $n \le 5$ 時這個猜測已經由張鎮華及阮夙姿証明它是正確的，参考資料在猜測的後面。

Conjecture (G.J. Chang and F.K. Hwang)

A bipartite graph with 2^n edges contains an induced subgraph of size 2^{n-1} for $n \ge 1$.

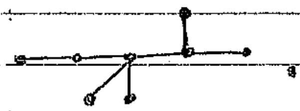
Reference

Group Testing in Bipartite Graphs, Vol. 6, 2002, Taiwanese J. Math., pp. 67-73.
(Su Juan and G. Chang)

(*) 如果 2^n 改成 $2 \cdot k$，就不一定找得到 Induced Subgraph of size k. 例如 $K_{5,9} - e$ 有 44 个边，它沒有 22 个边的 Induced subgraph.

Tree Packing problem

Given a tree T of size $\|T\| = n$. Can we pack $2n+1$ copies of T into the complete graph of order $2n+1$?

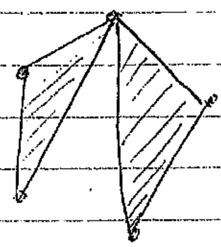
For example, T: 

Fact 1 K_{2n+1} has $(2n+1)\cdot n$ edges.

Fact 2 If we can pack T ($\|T\| = n$) into K_{2n+1}, then it is a decomposition of K_{2n+1} into T's.

Review An $\mathcal{H}$-decomposition of G is a partition of $E(G)$ such that each part induces a graph in $\mathcal{H}$. If $\mathcal{H} = \{H\}$, then we simply say G has an H-decomposition, denoted by $H \mid G$.

Review A packing or $\mathcal{H}$-packing of G is a collection of edge-disjoint subsets of $E(G)$ such that each part induces a graph in $\mathcal{H}$.

e.g. A K_3-packing of K_5. 

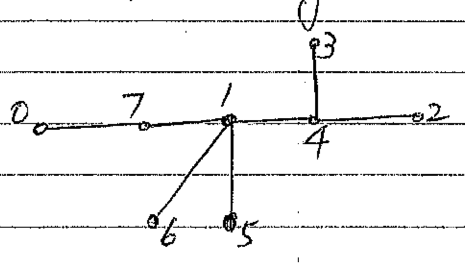
(*) Open problem

For which 3-sufficient graph G, G has a K_3-decomposition.

① $|G| \geq 3$, ②, $3 \mid \|G\|$ and ③ $\forall v \in V(G)$, $\deg_G(v)$ is even.

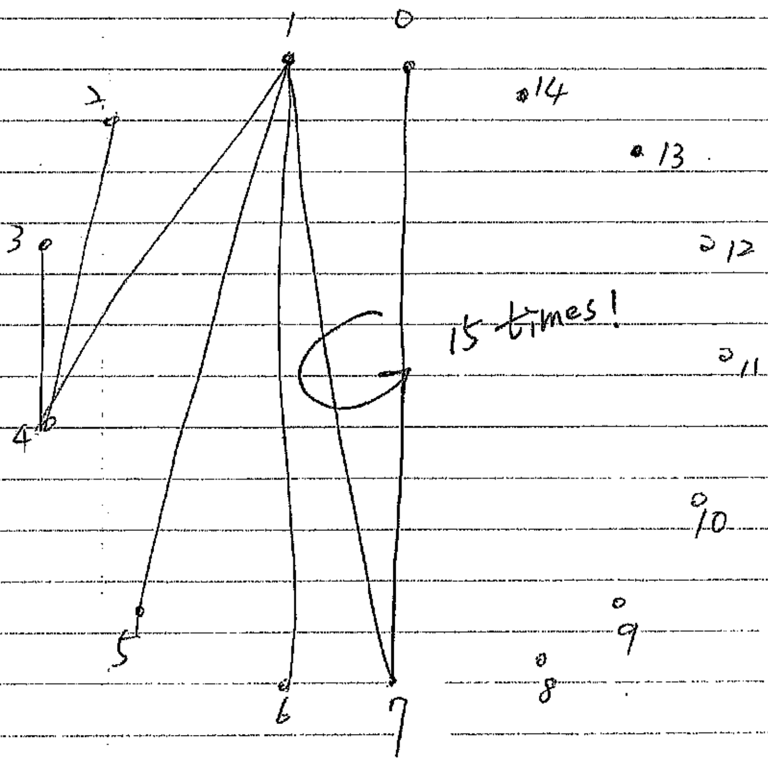
<u>Open problem</u> For any tree T of size n, prove that K_{2n+1} can be decomposed into copies of T, i.e., $T \mid K_{2n+1}$.

An idea of packing.



(Find a good labeling!)

[Graceful labeling or β-labeling]



<u>Definition</u> (Graceful labeling) (Or a β-labeling)

A vertex-labeling $\varphi : V(G) \xrightarrow{1-1} \{0, 1, 2, \cdots, \|G\|\}$ is a graceful labeling if all the weights of uv, $|\varphi(u) - \varphi(v)|$ are distinct.

<u>Graceful labeling tree conjecture</u>: Trees are graceful! ($750,-)
(P. Erdős)

$\kappa(G)$: Connectivity of G , $\kappa'(G)$: edge-connectivity of G

Theorem 14

For each connected graph G; $\kappa(G) \underset{①}{\le} \kappa'(G) \underset{②}{\le} \delta(G)$.

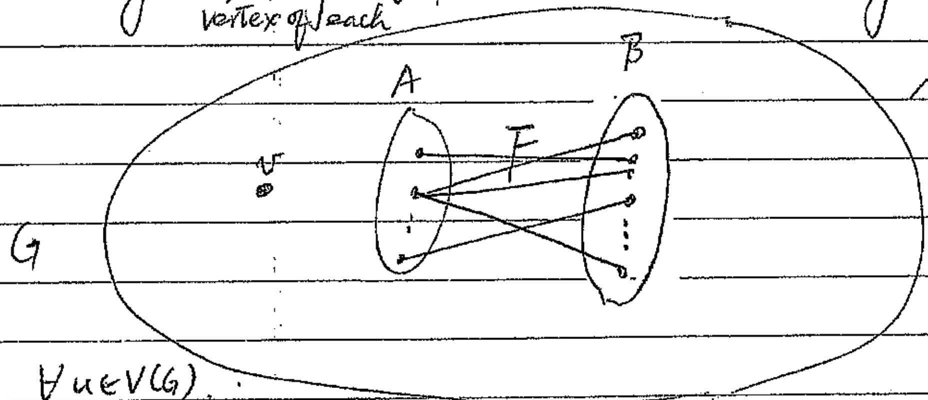
Proof.

② Let v be a vertex of G such that $\deg_G(v) = \delta(G)$. Since the deletion of all edges incident v will provide a disconnected graph, $\delta(G) \ge \kappa'(G)$.

① Let F (Figure 14) be an edge cut of G which has $\kappa'(G)$ edges.

Case 1. $\exists\, v \in V(G)$, v is not incident any edge of F.

Let C be the component in $G-F$ which contains v. Then, C contains exactly one vertex of each edge of F. (?) Hence, deleting all such vertices results in a disconnected graph. Hence $\kappa(G) \le \kappa'(G)$.



Case 2. $\forall u \in V(G)$, u is incident to an edge of F. (next page)

Figure 14. Edge-cut F

(°) It is interesting to know the graphs G satisfying $\kappa(G) = \delta(G)$.

Let C be the component of $G-F$ which contains an arbitrary vertex v. Now, if $wv \notin F$, then w is incident to an edge of F. Then, $\deg_G(v) \le |F|$. This implies (again) $\kappa(G) \le |F| = \kappa'(G)$. On the other hand all, $wv \in F$, $w \in N_G(v)$, then we have a complete graph of order $|F|+1 = |G|$. (W.L.O.G.) $\kappa(G) = |G|-1 = |F| = \kappa'(G)$. ∎

<u>Theorem 15</u> (Mader, 1972)

Let $k \in \mathbb{N}$. Every graph G with $d(G) \ge 4k$ has a $(k+1)$-connected subgraph H such that $\varepsilon(H) > \varepsilon(G) - k$.

<u>Proof.</u> For convenience, let $\varepsilon = \varepsilon(G)$ $\geq 2k$ ($d(G) \ge 4k$) and consider $G' \le G$ s.t. $|G'| \ge 2k$ and $\|G'\| > \varepsilon \cdot (|G'| - k)$. Since $\|G\| = \varepsilon \cdot |G| > \varepsilon(|G| - k)$, such graphs G' do exist. <u>Let H be the one with minimum order.</u>

Clearly, $|H| > 2k$, for otherwise, $\|H\| \ge \varepsilon \cdot k \ge 2k^2 > \binom{|H|}{2} = k \cdot (2k-1)$. ($|H| = 2k$)

The minimality of H implies that $\delta(H) > \varepsilon$, by Theorem 13 (we can choose a proper induced subgraph). Hence, $|H| \ge \varepsilon$. By the choice of H, we have $\|H\| > \varepsilon \cdot (|H| - k)$ mentioned above, thus $\varepsilon(H) > \varepsilon - k$.

$$\left(\frac{\|H\|}{|H|} > \frac{\varepsilon(|H|) - \varepsilon \cdot k}{|H|} = \varepsilon - \frac{\varepsilon k}{|H|} \ge \varepsilon - \frac{|H| \cdot k}{|H|} = \varepsilon - k \right)$$

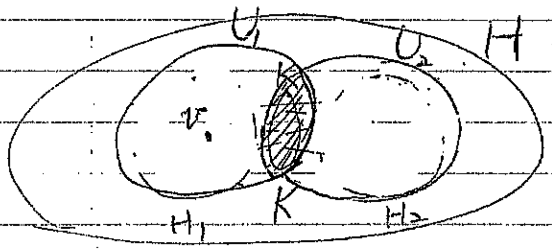
If $\delta(H) \le \varepsilon$, let v' be a vertex with $\deg_H(v') \le \varepsilon$. $H' = H - v'$ is the one with smaller order, $\rightarrow\leftarrow$. (平均边數沒有变小)

Therefore, H satisfies the condition we need. It's left to check that H is $(k+1)$-connected. Suppose not. Let K be a set of k vertices in H such that $H-K$ is disconnected, see Figure 15.

Now, let $V(H) = U_1 \cup U_2$ such that $U_1 \cap U_2 = K$, $H_1 = \langle U_1 \rangle_H$, $H_2 = \langle U_2 \rangle_H$ and no vertices of $U_1 \setminus U_2$ is adjacent to vertices in $U_2 \setminus U_1$.

Let $v \in U_1 \setminus U_2$. Since $\deg_H(v) \geq \delta(H) > \varepsilon$, this implies that $|H_1| \geq \varepsilon \geq 2k$, so is $|H_2| \geq 2k$. Because of the fact that H is the choice of G' with minimum order, $\|H_1\| \leq \varepsilon(|H_1| - k)$ and $\|H_2\| \leq \varepsilon(|H_2| - k)$. Hence $\|H\| \leq \|H_1\| + \|H_2\| \leq \varepsilon(|H_1| + |H_2| - 2k) = \varepsilon(|H| - k)$.

A contradiction to the choice of H.



$U_1 \cap U_2 = K$

Figure 15, $H_1 = \langle U_1 \rangle_H$ and $H_2 = \langle U_2 \rangle$