Theorem 1 – 5 (Lecture 2) Sep. 21, 22, 2022

Theorem 1 (Veblen, 1912)

The edge set of a graph can be partitioned into cycles if and only if every vertex has even degree.

Proof. ($\Rightarrow$) A vertex contained in t cycles has degree $2t$.

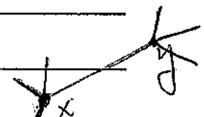
($\Leftarrow$) The cycles can be obtained recursively. We start with finding the first cycle. Let $\langle x_0, x_1, \cdots, x_\ell \rangle$ be a path of maximum length ℓ in G. Since $x_0x_1 \in E(G)$, $\deg_G(x_0) \geq 2$. Let $y (\neq x_1)$ be a neighbor of x_0, i.e., $x_0 y \in E(G)$. Now, $y \in \{x_2, x_3, \cdots, x_\ell\}$. For otherwise, we have a longer path. So, if $y = x_i$, then we have a cycle $C = (x_0, x_1, \cdots, x_i)$. The process continues in $G - E(C)$. (Each vertex is of even degree in the graph $G - E(C)$.) ∎

Theorem 2 (Mantel, 1907)

Every graph of order n and size greater than $\lfloor \frac{n^2}{4} \rfloor$ contains a triangle (C_3 or K_3). Proof. (1st)

Since $K_3 \nsubseteq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \phi$. This implies that $\deg_G(x) + \deg_G(y) \le |G| = n$. (Figure 1) Now, consider

$$\sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) = \sum_{x \in V(G)} (\deg_G(x))^2 \quad \text{(Two-way counting)}.$$



相邻的 deg(x)
边都算一次 deg(x)

$$\le n \cdot \|G\| = n \cdot e(G)$$

By Cauchy's inequality, $(2e(G))^2 = \left(\sum_{x \in V(G)} \deg_G(x)\right)^2 \le n \cdot \sum_{x \in V(G)} (\deg_G(x))^2$

$$\le n^2 \cdot e(G).$$

Hence, $e(G) \le \frac{n^2}{4}$. ▨

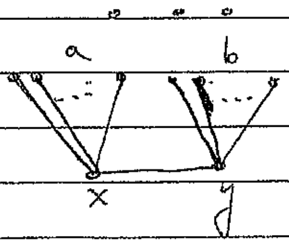


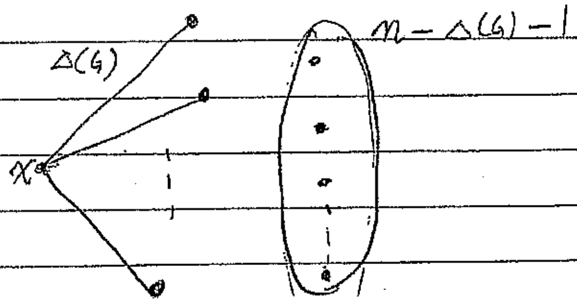
Figure 1. $\deg_G(x) = a+1$, $\deg_G(y) = b+1$

(2nd proof.)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2)

Since $K_3 \nsubseteq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies that $\|G\| \le \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G))$.

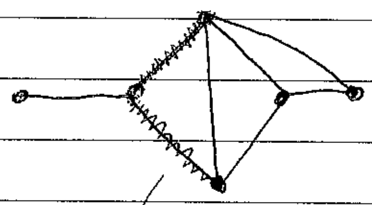
$\|G\|$ will take a maximum when $\Delta(G) = \lfloor \frac{n}{2} \rfloor$. Hence, we have the proof. ▨



(Original form) denoted by $H \leq G$

(∘) H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
(See 2'' for general version.)

(∘) H is an induced subgraph of G denoted by $H \lesssim G$ if for each $uv \in E(G)$ and $u, v \in V(H)$, then $uv \in E(H)$. We use $\langle S \rangle_G$ to denote the induced subgraph induced by $S \subseteq V(G)$. Hence, if $H = \langle S \rangle_G$, then $H = \langle V(H) \rangle_G$.



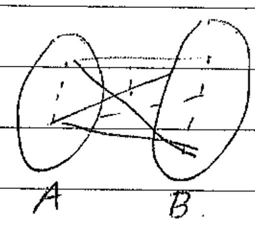
A subgraph, but not an induced subgraph

(∘) P_m : path with m vertices (order m)

C_ℓ : cycle with ℓ vertices (order ℓ)

K_n : complete graph of order n

K_{n_1,n_2} : complete bipartite graph (A,B)



(∘) If we would like to find a graph G of order n which does not contain a subgraph F, then F is called a forbidden graph of G. The maximum size of G is denoted by $ex(n;F)$. Thm. 2 $\Rightarrow$ $ex(n;C_3) \leq \lfloor \frac{n^2}{4} \rfloor$.

(*) $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is a graph of size $\lfloor \frac{n^2}{4} \rfloor$ which contains no C_3's. Hence, $ex(n; C_3) = \lfloor \frac{n^2}{4} \rfloor$.

(•) The graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is an extremal graph which forbids C_3 (or K_3).

(***) Find $ex(n; C_4)$. (Or $ex(n; C_k)$ for $k \geq 4$.)

Unsolved

Note. For certain n, $ex(n; C_4)$ is known.

Ex. 1-3. Find $ex(101; K_{11})$.

(•) Two graphs G_1 and G_2 are isomorphic if there exists a bijection $\varphi : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ iff $\varphi(u)\varphi(v) \in E(G_2)$, equivalently, $u \sim_{G_1} v \Leftrightarrow \varphi(u) \sim_{G_2} \varphi(v)$.

(:) A graph H is a subgraph (general form) of G if H is isomorphic to a subgraph (original form) of G. (長相一樣即可)

(•) φ is called an automorphism of G if $G_1 = G_2 = G$.

(*) Let G be a graph. The set of all automorphisms of G is denoted by $Aut(G)$. ($\langle Aut(G), \circ \rangle$ is a group.)

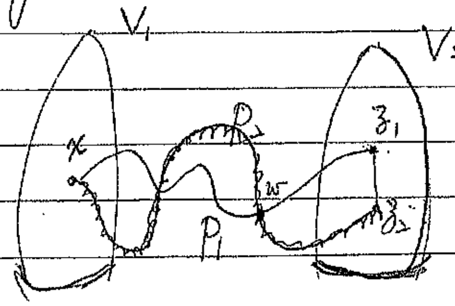
(*) $Aut(G)$ can be used to characterize the structure of G.

Theorem 3 A graph is bipartite if and only if it does not contain an odd cycle.

Proof. ($\Rightarrow$) Let $G = (A, B)$ where A and B are its partite sets. If $(x_0, x_1, \cdots, x_\ell)$ is a cycle of G, then x_0 and x_ℓ must in different partite sets. Hence, the index ℓ must be odd, thus the cycle is of even length.

($\Leftarrow$) W.L.O.G., let G be a connected graph. Let $x \in V(G)$ and $V_1 = \{y \mid y \in V(G) \text{ and } d(x,y) \text{ is even}\}$. Hence, $x \in V_1$. Let $V_2 = V(G) \setminus V_1$. It suffices to claim that both V_1 and V_2 are independent sets. First, consider V_2. Clearly, for each $z \in V_2$, $d(x,z)$ is odd. Suppose that $z_1, z_2 \in V_2$ and $z_1 \sim z_2$ ($z_1z_2 \in E(G)$). (Figure 3) Let P_1 and P_2 be the two paths such that $P_1 = \langle x, \cdots, z_1 \rangle$ and $P_2 = \langle x, \cdots, z_2 \rangle$, moreover they are the shortest paths connecting x to z_1 and x to z_2 respectively.

Figure 3



Let w be the last vertex in which P_1 and P_2 intersect. Also, let $\|P_1\| = 2s+1$ and $\|P_2\| = 2t+1$. (Note that if $V(P_1) \cap V(P_2) = \{x\}$, then we have an odd cycle (x, P_1, z_1, z_2, P_2) (length $2s+2t+3$).) Now, if w does exist, then $\langle x, P_1, w\rangle$ and $\langle x, P_2, w\rangle$ are of the same length, let the length be h. (?) So, the cycle $(w, \cdots, z_1, z_2, \cdots)$ is of length $(2s+1-h)+(2t+1-h)+1 = 2s+2t-2h+3$, an odd integer. Thus, an odd cycle exists, a contradiction. Hence, V_2 is an independent set. A similar argument can be applied to show that V_1 is also an independent set. (x is not adjacent to any vertex of $V_1 \setminus \{x\}$.) Figure 4.

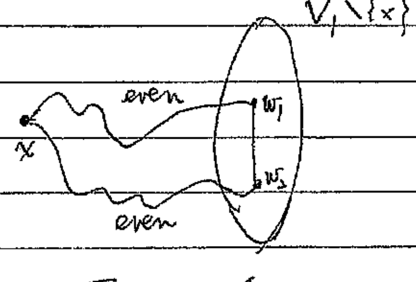


Figure 4.

<u>Open problem</u> How many edges can a bipartite graph of partite sets A, B, $|A|=m$, $|B|=n$, have such that $G \not\supseteq C_4$?

The maximum size is denoted by $z(m,n;2,2)$.

__Theorem 4__ The following statements are equivalent for a graph G.

(a) G is a tree. (G is connected and acyclic.)

(b) G is connected and every edge of G is a bridge.

(c) G is a maximal acyclic graph. (If x and y are not adjacent, then $G+xy$ contains a cycle.

__Proof.__ (a) $\Rightarrow$ (b)

Let xy be an edge of G and $G-xy$ is connected. Then, there exists a path P connecting x and y in $G-xy$. Clearly, G contains a cycle (x, P, y) in G, a contradiction.

(b) $\Rightarrow$ (c) If G is not acycle, then a cycle edge is not a bridge. Hence, G is acyclic.

If G is not a maximal acyclic graph, then there exists a pair of vertices z_1 and z_2 in G such that $G+z_1z_2$ is also acyclic. Since G is connected, there exists a path connecting z_1 and z_2, say P. This implies that (z_1, P, z_2) is a cycle in $G+z_1z_2$, a contradiction.

(c) $\Rightarrow$ (a) If G is not connected, then there exists a pair of vertices x_1 and x_2 such that $G+x_1x_2$ is also acyclic.
(in different components)

x_1 and x_2

Theorem 5 (Euler, 1941)

A ~~nontrivial~~ connected graph (multigraph) has an eulerian circuit (Euler circuit) if and only if each vertex has even degree. Moreover, a connected graph has an eulerian trail from a vertex x to a vertex $y \neq x$ if and only if x and y are the only two vertices of odd degree.

Proof. The second statement follows directly from the first one. We prove the first statement.

($\Rightarrow$) If a circuit passes a vertex x h times, then $\deg_G(x) = 2h$.

($\Leftarrow$) By induction on $\|G\|$. Since $\|G\| \geq 1$, $\delta(G) \geq 2$ and thus G contains a cycle. (G is not a tree!)

Let Z be a circuit in G with the maximum number of edges. If Z is an eulerian circuit, then we are done. Suppose not. Let H be a nontrivial component of $G - E(Z)$. Since G is connected, $V(H) \cap V(Z) \neq \phi$. Let $x \in V(H) \cap V(Z)$. Now, H is a nontrivial connected graph (even graph). Hence, H contains an eulerian circuit Y (by induction). By using x, we can attach

Z and Y together to obtain a larger circuit. (Figure 5) This contradicts to the maximality of $|E(Z)|$. Hence, Z must be an eulerian circuit of G. ■

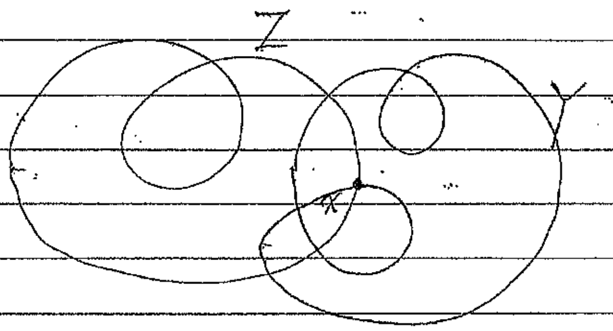
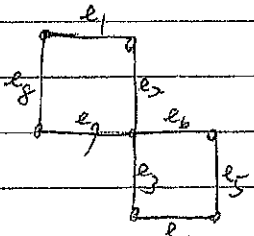


Figure 5. Attaching Z and Y.

Open problem

Find the number of distinct eulerian circuits of an eulerian graph G. (Two circuits are the same if they can be obtained each other by a cyclic shift of edges.)



$$\langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \rangle$$

$$=_{def} \langle e_3, e_4, e_5, e_6, e_7, e_8, e_1, e_2 \rangle$$

$$\neq_{def} \langle e_1, e_2, e_6, e_5, e_4, e_3, e_7, e_8 \rangle$$

Theorem 5.5 (BEST Theorem)

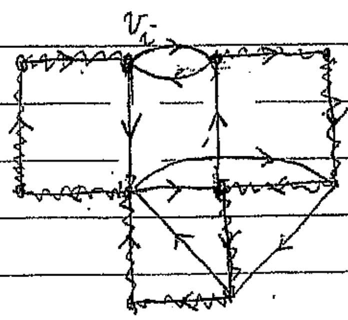
A digraph D has an eulerian (directed) circuit if and only if D is strongly connected and for each vertex $v \in V(D)$, $\deg^+_D(v) = \deg^-_D(v)$. Moreover, if D is an eulerian graph, $s(D)$ is the number of distinct eulerian circuits, then

$$s(D) = t_i(D) \cdot \prod_{j=1}^{n} (\deg^+_D(v_j) - 1)! \quad \text{for every } i \in \{1, 2, \dots, n\}$$

where $t_i(D)$ is the number of spanning trees oriented toward v_i.

(Note: The counting part of the theorem was proved by de Bruijn and van Aardenne-Ehrenfest (independently) two groups Smith and Tutte.)

Proof. The existence part can be obtained by a similar argument as the "multigraph" version.

D: 

$s(D) = 1 \cdot 1 = 1$

Figure 6 Spanning tree oriented toward v_i