Lecture 9 Nov. 30

Date / / No. 1

1. Introduction and preliminaries

⋮ See (1′) ~ (2′)

<u>Definition 1.0.</u> Let $A_1, A_2, \cdots, A_m$ be events in an arbitrary probability space. A graph $G=(V,E)$ on the set of vertices $V=\{1,2,\cdots,m\}$ is said to be a dependency graph for the events $A_1, A_2, \cdots, A_m$ if for each i, $1 \le i \le m$, the event A_i is mutually independent of a set of all the other events except for those A_j with $\{i,j\} \in E$.

In the following, e denotes the base of natural logrithms, i.e., $e \approx 2.71828$.

<u>Theorem 1.1.</u> (The Lovász Local Lemma; Symmetric case)

Let $A_1, A_2, \cdots, A_m$ be events in an arbitrary probability space. Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most μ, and that $Pr(A_i) \le p$ for all $1 \le i \le m$. If $e \cdot p \cdot (\mu+1) \le 1$, then $Pr(\bigcap_{i=1}^{m} \overline{A_i}) > 0$.

(*) You may consider A_i as "bad" event. Then, $\overline{A_i}$ is a good event. (Good events 同時發生的機率大於 0 !)

Definition 7.1.1 (Discrete Probabilistic Space) , D.P.S.

A D.P.S. is an ordered pair (S, f) where S is a countable set and $f : S \to \mathbb{R}$ satisfying (i) $0 \leq f(x) \leq 1$ and (ii) $\Sigma_{x \in S} f(x) = 1$.

(Note) A countable set is either a finite set or an infinite set which has the same cardinality as $\mathbb{N}$.

Definition 7.1.2 (The probability of an event $A \subseteq S$)

Let (S, f) be a D.P.S.. Then the probability of $A \subseteq S$ is $P(A) = \Sigma_{x \in A} f(x)$.

Definition 7.1.3 (Independent events)

If $P(A \cap B) = P(A)P(B)$, then A and B are independent events.

Definition 7.1.4 (Random variables)

Let (S, f) be a D.P.S.. Then $\mathbb{X} : S \to \mathbb{R}$ is a random variable where we use $\mathbb{X} = k$ to denote an event.

$$K = \{x \in S \mid \mathbb{X}(x) = k\}.$$

e.g. Let $S = [1, 6]^2$ and $f(x, y) = \frac{1}{36}$ for each $(x, y) \in [1, 6]^2$. $\mathbb{X}((x, y)) = x + y$, $k = 7 \Rightarrow (\mathbb{X} = 7) = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$.

Definition 7.1.5 (Expectation)

Let $\mathbb{X}$ be a random variable. Then the expectation of $\mathbb{X}$, $\mathbb{E}(X) = \Sigma_k k \cdot p(\mathbb{X} = k)$. (We define $P(\mathbb{X} = h) = 0$ if h is not in the image of $\mathbb{X} : S \to \mathbb{R}$.)

e.g. (Continued), $\mathbb{X} = 7$.

$$\begin{aligned}
\mathbb{E}(X) &= 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{12} + 5 \cdot \frac{1}{9} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{1}{6} \\
&+ 12 \cdot \frac{1}{36} + 11 \cdot \frac{1}{18} + 10 \cdot \frac{1}{12} + 9 \cdot \frac{1}{9} + 8 \cdot \frac{5}{36} \\
&= 14 \cdot (\frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{1}{9} + \frac{5}{36} + \frac{1}{12}) \\
&= 14 \cdot \frac{1 + 2 + 3 + 4 + 5 + 3}{36} = 7.
\end{aligned}$$

Lemma 7.1.1 (Pigeon-hole principle of Expectation)

Let $\mathbb{X}$ be a random variable of a D.P.S. Then, there exists a $y \in S$ such that $\mathbb{X}(y) \geq \mathbb{E}(\mathbb{X})$.

Lemma 7.1.2 (Linear Property of Expectation)

Let $X, X_1, \cdots, X_m$ be random variables such that $X = \Sigma_{i=1}^{m} X_i$. Then, $\mathbb{E}(X) = \Sigma_{i=1}^{m}\mathbb{E}(X_i)$.

Definition 7.1.6 (Indicstor Random Variable)

An indicator random variable is a random variable X such that $\mathbb{X} : S \to \{0,1\}$ (instead if $\mathbb{R}$).

(Note) A random variable $\mathbb{X}$ can be written as a sum of $|G|$ indicator random variables

$$x_v = \begin{cases} 1, & \text{if } v \in \mathbb{X} \text{ , and} \\ 0, & \text{otherwise} \end{cases}$$

Theorem 7.1.3 If $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} \leq 1$, then $R(k,k) > n$. Thus, $R(k,k) > \lfloor 2^{\frac{k}{2}} \rfloor$ for all $k \geq 3$.

Proof. Consider a random red-blue coloring of the edges of K_n. For a fixed set T of k vertices, let A_T be the event that $< T >$ is monochromatic. Hence, $P(A_T) = (\frac{1}{2})^{\binom{k}{2}} \cdot 2 = 2^{1-\binom{k}{2}}$. Since there are $\binom{n}{k}$ possible sets for T, the probability that at least one of the events A_T occurs is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$. By assumption, $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$. This implies that no event A_T occurs is of positive probability, i.e., there exists a red-blue coloring such that no monochromatic K_k exists, we have $R(k,k) > n$. Now, if we take $n = \lfloor 2^{\binom{k}{2}} \rfloor$,

$$\begin{aligned} &\binom{n}{k} 2^{1-\binom{k}{2}} \quad \text{where } 1 - \frac{k(k-1)}{2} = 1 - \frac{k^2}{2} + \frac{k}{2} \\ &< \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \\ &\leq \frac{(2^{\frac{k}{2}})^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} < 1. (k \geq 3) \end{aligned}$$

Hence, $R(k,k) > \lfloor 2^{\frac{k}{2}} \rfloor$ for all $k \geq 3$. This concludes the proof. □

Theorem 7.1.4 (Szele, 1943)

There exists a tournament T_n such that T_n has at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

2. The main results

For positive integers n and d, let $[n]$ denote the set $\{1, 2, \cdots, n\}$ and $\binom{[n]}{d}$ denote the collection of all d-subsets of $[n]$. Let $t(d,n)$ denote the minimum number of rows for a d-disjunct matrix with n columns. In [17], Yeh (葉鴻國) proved the following theorem by using Theorem 1.1.

Theorem 2.1. $t(d,n) \le d^{-d} \cdot (d+1)^{d+1} \cdot \left(1 + \ln(d+1) + \ln\left[\binom{n}{d+1} - \binom{n-d-1}{d+1}\right]\right)$.

To generalize Theorem 2.1, we start by giving a more general definition.

$(d, r; 1]$

Definition 2.2. A $t \times n$ binary matrix M is called $(d, r]$-disjunct if the union of any d columns does not contain the intersection of any other r columns in M.

It is worth of noting that a $(d, r]$-disjunct matrix can identify the up-to-d positives on the complex model with complexes of size at most r, see [3] for a reference. (Lecture 8)

Let $t(n, d, r]$ denote the minimum number of rows for a $(d,r]$-disjunct matrix with n columns. Then we have the following generalization of Theorem 2.1.

<u>Theorem 2.3.</u> $t(n,d,r] \le d^{-d}\cdot r^{-r}\cdot(d+r)^{d+r}\cdot\left(1+\ln\left[\binom{n}{d}\binom{n-d}{r} - \binom{n-d-r}{d}\binom{n-2d-r}{r}\right]\right)$.

<u>Proof.</u> Let $M=(m_{i,j})$ be a $t\times n$ random binary matrix with $Pr(m_{i,j}=1)=p$ and $Pr(m_{i,j}=0)=1-p$. Note that all $m_{i,j}$'s are chosen independently. Let $A_{D,R}$ be the event that $\underline{\cap R \subseteq \cup D}$ where $D\in\binom{[n]}{d}$ and $R\in\binom{[n]}{r}$ with $D\cap R=\phi$. Then,

$$Pr(A_{D,R}) = (1-\underbrace{p^r\cdot(1-p)^d}_{t\times n})^t.$$ (R中有一列出現r個1，則D中那一列全部是0.)

By Lovász's Local Lemma, a $(d,r]$-disjunct matrix exists whenever
$e\cdot(\mu+1)\cdot Pr(A_{D,R}) \le 1$
where $\mu+1 = \binom{n}{d}\cdot\binom{n-d}{r} - \binom{n-(d+r)}{d}\cdot\binom{n-2d-r}{r}$. Here, μ is the maximum
(除了自己及相鄰的event)
degree of dependency graph.

Hence $e\cdot(1-p^r\cdot(1-p)^d)^t\cdot\left[\binom{n}{d}\cdot\binom{n-d}{r} - \binom{n-d-r}{d}\binom{n-2d-r}{r}\right] \le 1$ is required. This implies that

$$t \ge \frac{1+\ln\left[\binom{n}{d}\binom{n-d}{r} - \binom{n-d-r}{d}\binom{n-2d-r}{r}\right]}{-\ln(1-p^r\cdot(1-p)^d)}, \cdots (\#)$$

Since $-\ln(1-x) \geq x$ for $0 \leq x < 1$, we conclude that if

$$t \geq \frac{1 + \ln\left[\binom{n}{d}\binom{n-d}{r} - \binom{n-d-r}{d}\cdot\binom{n-2d-r}{r}\right]}{p^r \cdot (1-p)^d} \quad (2)$$

holds, then the inequality (*) holds. By the fact that $p = \frac{r}{d+r}$ gives the minimum value of R.H.S. of (2), we conclude the proof by pluging in p. ▨

By a similar technique, we also have the following result. First, we need a definition.

<u>Definition 2.4.</u> A $t \times n$ binary matrix M is called (d,r)-disjunct if the union of any d columns does not contain the union of any other r columns in M.

Note that an (h,d)-disjunct matrix can be applied to identify the positives on the (d,h)-inhibitor model. [6] 余國立論文 Here, h is the number of inhibitors.

Let $t(n,d,r)$ denote the minimum number of rows for a (d,r)-disjunct matrix with n columns. The following is also a generalization of Theorem 2.1.

Theorem 2.5. $t(n,d,r) \le \left(1+\frac{d}{r}\right)\left(1+\frac{r}{d}\right)^{\frac{d}{r}} \cdot \left(1+\ln\left[\binom{n}{d}\binom{n-d}{r}-\binom{n-d-r}{d}\binom{n-2d-r}{r}\right]\right)$

Proof. By a similar set up as in Theorem 2.3. Let $A_{D,R}$ be the event that $\cup R \subseteq \cup D$ where $D \in \binom{[n]}{d}$, $R \in \binom{[n]}{r}$ and $D \cap R = \phi$. Then, $Pr(A_{D,R}) = \left(1-(1-p)^d \cdot [1-(1-p)^r]\right)^t$. By using the Lovász Local lemma, we conclude that a $t \times n$ (d,r)-disjunct matrix exists whenever

$e \cdot Pr(A_{D,R}) \cdot \left[\binom{n}{d}\binom{n-d}{r} - \binom{n-d-r}{d}\binom{n-2d-r}{r}\right] \le 1$. Hence, we conclude the proof by letting $p = 1-\left(\frac{d}{d+r}\right)^{\frac{1}{r}}$. ▨

We can use the following notion to extend the above two types of disjunct matrices.

Definition 2.6. For $1 \le s \le r$, a $t \times n$ binary matrix M is called $(d, s \text{ out of } r]$-disjunct if for any d columns and any other r columns of M, there exists a row index in which none of the d columns appear and at least s of the r columns do.

Clearly, $(d, 1 \text{ out of } r]$-disjunctness is precisely the (d,r)-disjunctness and $(d, r \text{ out of } r]$-disjunctness is equivalent to $(d,r]$-disjunctness.

Now, with the same technique, we can also find a good upper bound for a the number of rows "t" in a $t \times n$ $(d, s \text{ out of } r]$-disjunct matrix.

Theorem 2.7. Let $t(n,d,r,s]$ denote the minimum number of rows we need in a $(d, s \text{ out of } r]$-disjunct matrix with n columns. Then, $t(n,d,r,s] \le \left(1+\ln\left[\binom{n}{d}\binom{n-d}{r}-\binom{n-d-r}{d}\binom{n-2d-r}{r}\right]\right)\Big/ f_{d,r,s}(p)$ for all $0<p<1$, where $f_{d,r,s}(p) = (1-p)^d \cdot \left[1-\sum_{i=0}^{s-1}\binom{r}{i}p^i(1-p)^{r-i}\right]$.

Proof. By using the same set up as above, let $\overline{A_{D,R}}$ be the event that there exists a row index in which none of the column $C_j, j \in D$, appear and at least s of the columns $C_k, k \in R$, do. Then,

$$\Pr(A_{D,R}) = \left\{1-(1-p)^d \cdot \left[1-\sum_{i=0}^{s-1}\binom{r}{i}p^i \cdot (1-p)^{r-i}\right]\right\}^t.$$

For convenience, let $f_{d,r,s}(p) = (1-p)^d \cdot \left[1-\sum_{i=0}^{s-1}\binom{r}{i}p^i \cdot (1-p)^{r-i}\right]$.

Again, by Theorem 1.1, a $t \times n$ $(d, s \text{ out of } r]$-disjunct matrix exists whenever $e \cdot [1-f_{d,r,s}(p)]^t \cdot \left[\binom{n}{d}\binom{n-d}{r}-\binom{n-d-r}{d}\binom{n-2d-r}{r}\right] \le 1$.

Hence, $t \ge 1+\ln\left[\binom{n}{d}\binom{n-d}{r}-\binom{n-d-r}{d}\binom{n-2d-r}{r}\right]\Big/ f_{d,r,s}(p)$ by using the fact $-\ln(1-x) \ge x$ for $0 \le x < 1$.

We can use similar approaches to obtain an upper bound for the minimum size (of rows) of a (k,m,n)-selector. Review that a (k,m,n)-selector with t rows is a $t\times n$ binary matrix M such that any submatrix of M obtained by choosing arbitrary k columns of M contains at least m rows of the identity matrix I_k. The integer t is commonly known as the size of the (k,m,n)-selector. Du and Hwang [9] proved that a (k,m,n)-selector is $(m-1, k-m+1)$-disjunct, but in general the reverse statement is not true. The (k,m,n)-selector does have quite a few applications in group testing, see [5] for example.

Let $t_s(k,m,n)$ denote the minimum size of a (k,m,n)-selector. De Bonis et. al. [5] obtained upper bounds for $t_s(k,m,n)$ by translating the problem into the hypergraph language.

Theorem 2.8. [5] $t_s(k,m,n) < \frac{ek^2}{k-m+1}\left(\ln\frac{n}{k}\right) + \frac{ek(2k-1)}{k-m+1}$.

The following upper bound is obtained by applying Theorem 1.1.

Theorem 2.9. $t_s(k,m,n) \le \frac{m}{\binom{k}{m}\cdot m!}\cdot\left[k\left(1+\frac{1}{k-1}\right)^{k-1}\right]^m\cdot\left[1+\ln\left(\binom{n}{k}-\binom{n-k}{k}\right)\right]$.

Date / /

Proof. Let $M^* = (m^*_{i,j})$ be a $t \times n$ random binary matrix with $Pr(m^*_{i,j} = 1) = p$ and $Pr(m^*_{i,j} = 0) = 1-p$. For $K \in \binom{[n]}{k}$ and $M \in \binom{[t]}{m}$, define A_K be the event that the $t \times k$ submatrix of M^* corresponding to K contains at most $m-1$ rows of I_k, and $A_{K,M}$ be the event that the $m \times k$ submatrix of M^* corresponding to K and M doesn't consist m distinct rows of I_k. Observe that $A_K = \bigcap_{M \in \binom{[t]}{m}} A_{K,M}$.

For convenience, assume that $m | t$. Let $M_i = \{m(i-1)+1, m(i-1)+2, \cdots, mi\}$ for $1 \le i \le \frac{t}{m}$. Then

$$Pr(A_K) = Pr\left(\bigcap_{M \in \binom{[t]}{m}} A_{K,M}\right) \le Pr\left(\bigcap_{i=1}^{t/m} A_{K,M_i}\right)$$

$$= \left[1 - \binom{k}{m} m! \cdot p^m \cdot (1-p)^{m(k-1)}\right]^{t/m}.$$

Note that A_K is mutually independent of all the other events $A_{K'}$ except for those $K' \cap K \ne \emptyset$. There are exactly $\binom{n}{k} - \binom{n-k}{k} - 1$ such events. Now, by Theorem 1.1, a $t \times n$ (k,m,n)-selector exists whenever

$$e \cdot \left[1-\binom{k}{m} m!\, p^m (1-p)^{m(k-1)}\right]^{\frac{t}{m}} \cdot \left[\binom{n}{k}-\binom{n-k}{k}\right] \le 1 .$$

This implies that

$$t \ge m \cdot \frac{1+\ln\left[\binom{n}{k}-\binom{n-k}{k}\right]}{-\ln\left[1-\binom{k}{m} m!\, p^m (1-p)^{m(k-1)}\right]} \quad \text{and}$$

thus $$t \ge m \cdot \frac{1+\ln\left[\binom{n}{k}-\binom{n-k}{k}\right]}{\binom{k}{m} m!\, p^m (1-p)^{m(k-1)}} .$$

Now, by plugging in $p=\frac{1}{k}$, we obtain the desired upper bound for $t_s(k,m,n)$. ∎

Stanford University **Spring 2018**
Math 233A: Non-constructive methods in combinatorics **Instructor: Jan Vondrák**
Lecture date: January 10, 2018 **Original scribe: Apoorva Khare**

Lecture 2. The Lovász Local Lemma

2.1 Introduction and motivation

We start with the *Lovász Local Lemma*, a fundamental tool of the "probabilistic method" and a prototypical non-constructive argument in combinatorics — proving that a certain object exists without showing what it looks like. Often in applying the probabilistic method, one is trying to show that it is possible to avoid "bad events" $\mathcal{E}_1, \dots, \mathcal{E}_n$ with positive probability, or in other words,

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] > 0.$$

Here $\mathcal{E}_i$ are subsets of a probability space Ω (typically a finite set), and $\overline{\mathcal{E}}_i = \Omega \setminus \mathcal{E}_i$ denotes the complementary event for each i.

If $\sum_i \mathbb{P}[\mathcal{E}_i] < 1$, then the above inequality clearly follows, by the "union bound". However, this is often not a strong enough tool, since the sum $\sum_i \mathbb{P}[\mathcal{E}_i]$ may be much larger than 1 even if the events can be avoided.

A weaker constraint on the individual probabilities $\mathbb{P}[\mathcal{E}_i]$ is sufficient if the events $\mathcal{E}_i$ are also independent. In that case if $\mathbb{P}[\mathcal{E}_i] < 1$ for all i, then $\mathbb{P}[\cap_i \overline{\mathcal{E}}_i]$ is clearly positive. The Lovász Local Lemma is an effective refinement of this phenomenon, for events that do not have "too much (inter)dependency" – a notion that will be made precise presently. An additional attractive feature of the Local Lemma is that it does not place any restriction on the (finite) number of events $\mathcal{E}_i$.

2.2 Symmetric Local Lemma and application to hypergraph colorability

Before we state and prove the Local Lemma, we first present a prototypical application of the result, which serves to motivate it.

Example 2.1 (Hypergraph 2-coloring) *Given an integer $k \geq 2$, a* k-uniform hypergraph $G = (V(G), E(G))$ *consists of a finite set of nodes $V(G)$ and a collection of subsets $e_1, \dots, e_n \subset V(G)$, each of size k, which are termed "edges" (or "hyper-edges"). We want to color each node in $V(G)$ either red or blue. Under what conditions can we guarantee that there is a coloring with no monochromatic edge, i.e., every edge contains both red and blue nodes? Such hypergraphs are said to be* 2-colorable.

Notice, if we color each node red or blue uniformly at random, then the event $\mathcal{E}_i$ that the ith edge is monochromatic has probability 2^{1-k}. Thus if the hypergraph G has less than 2^{k-1} edges, then by the union bound, the probability that there is at least one monochromatic edge is $< 2^{k-1} \cdot 2^{1-k} = 1$. It follows that G is 2-colorable.

However, this argument fails when G has $\geq 2^{k-1}$ edges. In this case, under what assumptions can we prove 2-colorability? One such assumption is that every edge intersects at most d other

$2 \cdot \frac{1}{2^k}$

edges, for some d. Under such an assumption, we will show 2-colorability using the Lovász Local Lemma. (Interestingly, d will be comparable to 2^{k-1}.)

We now define the following notion of mutual independence.

Definition 2.2 *For all integers $n > 0$, define $[n] := \{1, \dots, n\}$. Given events $\mathcal{E}_1, \dots, \mathcal{E}_n \subset \Omega$ and a subset $J \subset [n]$, the event $\mathcal{E}_i$ is said to be* mutually independent *of $\{\mathcal{E}_j : j \in J\}$ if for all choices of disjoint subsets $J_1, J_2 \subset J$,*

$$\mathbb{P}\left[\mathcal{E}_i \cap \bigcap_{j_1 \in J_1} \mathcal{E}_{j_1} \cap \bigcap_{j_2 \in J_2} \overline{\mathcal{E}}_{j_2}\right] = \mathbb{P}[\mathcal{E}_i] \cdot \mathbb{P}\left[\bigcap_{j_1 \in J_1} \mathcal{E}_{j_1} \cap \bigcap_{j_2 \in J_2} \overline{\mathcal{E}}_{j_2}\right].$$

Equipped with this notion, we can state the first form of the Lovász Local Lemma, which will help answer the above question of 2-colorability for k-uniform hypergraphs.

Theorem 2.3 (Symmetric Lovász Local Lemma) *Suppose $p \in (0,1)$, $d \geq 1$, and $\mathcal{E}_1, \dots, \mathcal{E}_n$ are events such that $\mathbb{P}[\mathcal{E}_i] \leq p$ for all i. If each $\mathcal{E}_i$ is mutually independent of all but d other events $\mathcal{E}_j$, and $ep(d+1) \leq 1$, where $e = 2.71828\dots$ is Euler's number, then $\mathbb{P}\left[\cap_{i=1}^n \overline{\mathcal{E}}_i\right] > 0$.*

Remark 2.4 *In the above result, d is sometimes called the "dependence degree". The "local"-ness of the result has to do with the fact that assumptions depend only on d rather than n, the number of events.*

Before we prove the Local Lemma (in a more general form), let us see how it can be used to study 2-colorability for hypergraphs. In the setting of Example 2.1, suppose the hypergraph G has n edges, denoted by $e_1, \dots, e_n$. Let $\mathcal{E}_i$ denote the event that the edge e_i is monochromatic; as computed above, $p = 2^{1-k}$.

We now claim that "$d = d$", that is, the d in Example 2.1 is precisely the d in Theorem 2.3. Indeed, fix an edge e_i; now any conditioning (i.e., node-coloring) on the edges disjoint from e_i is independent of $\mathcal{E}_i$, since the node colors are i.i.d. Bernoulli random variables. Thus, the assumptions of the Symmetric Lovász Local Lemma are indeed satisfied, as long as

$$d + 1 \leq \frac{1}{ep} = \frac{2^{k-1}}{e}.$$

We stress again that this condition is independent of the number of edges in the hypergraph G.

2.3 (Asymmetric) Lovász Local Lemma: statement and proof

We now prove the Symmetric Lovász Local Lemma, i.e., Theorem 2.3. In fact we show a stronger, "asymmetric" version, and use it to prove the symmetric version. This will require the following useful concept.

Definition 2.5 *A (directed) graph $G = (V(G), E(G))$ is a* dependency (di)graph *on events $\mathcal{E}_1, \dots, \mathcal{E}_n$ if $V(G) = [n]$ and each event $\mathcal{E}_i$ is mutually independent of its non-neighbors $\{\mathcal{E}_j : j \neq i,\ (i,j) \notin E(G)\}$.*

Remark 2.6 *Most applications in the literature use the undirected version of the dependency graph; however, there are some applications that use the digraph structure. In such cases, given a directed edge (i, j), i is the source and j the target.*

We can now state the Lovász Local Lemma in its more general form.

Theorem 2.7 ((Asymmetric) Lovász Local Lemma) *Suppose G is a dependency (di)graph for events $\mathcal{E}_1, \ldots, \mathcal{E}_n$, and there exist $x_1, \ldots, x_n \in (0, 1)$ such that*

$$\mathbb{P}[\mathcal{E}_i] \leq x_i \prod_{(i,j)\in E(G)} (1 - x_j), \quad \forall i \in [n]. \tag{2.1}$$

Then,

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] \geq \prod_{i=1}^{n}(1 - x_i) > 0. \tag{2.2}$$

Remark 2.8 *Given a set of events $\mathcal{E}_i$, the choice of a dependency digraph G is not unique, nor is the choice of the parameters x_i. Rather, the "user" decides which dependency digraph G and parameters x_i to work with, in a given application. The dependency graph is often clear from the context (e.g. in the hypergraph colorability application above), although the choice of x_i might not be.*

Remark 2.9 *Theorem 2.7 is sharp when the $\mathcal{E}_i$ are independent, G is empty, and $x_i = \mathbb{P}[\mathcal{E}_i]$ $\forall i$.*

Before we show the Asymmetric Local Lemma, let us quickly see why it implies the Symmetric version. Indeed, if the hypotheses of Theorem 2.3 hold, set $x_i = \frac{1}{d+1}$ $\forall i$. Now the hypotheses imply that there is an undirected dependency graph G in which each node has degree at most d. Therefore,

$$\begin{aligned} x_i \prod_{(i,j)\in E(G)} (1 - x_j) = \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{\deg(i)} &\geq \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{d} \\ &\geq \frac{1}{d+1} \cdot \frac{1}{e} \geq \mathbb{P}[\mathcal{E}_i]. \end{aligned}$$

It follows by the Asymmetric Lovász Local Lemma that $\mathbb{P}[\cap_i \overline{\mathcal{E}}_i] > 0$.

Finally, we prove the Asymmetric Lovász Local Lemma.

Proof of Theorem 2.7. Given $S \subset [n]$, define

$$\overline{P}_S := \mathbb{P}\left[\bigcap_{i\in S} \overline{\mathcal{E}}_i\right], \qquad \overline{P}_\emptyset := 1.$$

The result follows once we show, by induction on $|S|$, that for all $S \subset [n]$ and $a \in S$,

$$\frac{\overline{P}_S}{\overline{P}_{S\setminus\{a\}}} \geq 1 - x_a. \tag{2.3}$$

More precisely, we will show by induction on $|S|$ that

$$\overline{P}_S \geq (1 - x_a)\overline{P}_{S\setminus\{a\}} > 0.$$

Indeed, this yields the result, because applying the inequality to $S = [n]$, then $[n-1]$, and so on, yields:

$$\mathbb{P}\left[\bigcap_{i=1}^{n} \overline{\mathcal{E}}_i\right] = \overline{P}_{[n]} \geq (1 - x_n)\overline{P}_{[n-1]} = (1 - x_n)(1 - x_{n-1})\overline{P}_{[n-2]} \geq \cdots \geq \prod_{i=1}^{n}(1 - x_i) > 0,$$

as desired.

Thus it remains to prove (2.3). The base case is when $S = \{a\}$ is a singleton. In this case,

$$\frac{\overline{P}_{\{a\}}}{\overline{P}_{\emptyset}} = \mathbb{P}[\overline{\mathcal{E}}_a] \geq 1 - x_a \prod_{(a,j)\in E(G)} (1 - x_j) \geq 1 - x_a,$$

proving the assertion. Now suppose (2.3) holds for all subsets $S' \subset [n]$ with size at most k, and say $S \subset [n]$ has size $k+1$. To proceed further, let us define the neighborhood of $a \in S$, as well as its "closure", via:

$$\Gamma(a) := \{j \in V(G) : (a,j) \in E(G)\}, \qquad \Gamma^+(a) := \{a\} \cup \Gamma(a). \tag{2.4}$$

Now fix $a \in S$, and compute:

$$\overline{P}_S = \mathbb{P}\left[\bigcap_{i\in S} \overline{\mathcal{E}}_i\right] = \mathbb{P}\left[\bigcap_{i\in S\setminus\{a\}} \overline{\mathcal{E}}_i\right] - \mathbb{P}\left[\mathcal{E}_a \cap \bigcap_{i\in S\setminus\{a\}} \overline{\mathcal{E}}_i\right] \geq \mathbb{P}\left[\bigcap_{i\in S\setminus\{a\}} \overline{\mathcal{E}}_i\right] - \mathbb{P}\left[\mathcal{E}_a \cap \bigcap_{i\in S\setminus\Gamma^+(a)} \overline{\mathcal{E}}_i\right]$$
$$= \overline{P}_{S\setminus\{a\}} - \mathbb{P}[\mathcal{E}_a]\overline{P}_{S\setminus\Gamma^+(a)},$$

where the first equality and the inequality are straightforward, and the final equality follows from the mutual independence of $\mathcal{E}_a$ and $\{\mathcal{E}_i : i \notin \Gamma^+(a)\}$. From this computation it follows that

$$\frac{\overline{P}_S}{\overline{P}_{S\setminus\{a\}}} \geq 1 - \mathbb{P}[\mathcal{E}_a]\frac{\overline{P}_{S\setminus\Gamma^+(a)}}{\overline{P}_{S\setminus\{a\}}},$$

where $\overline{P}_{S\setminus\{a\}} > 0$ by the induction hypothesis. ~~Now say $\Gamma(a) \cap S =$~~ $\{b_1, \dots, b_d\}$ for some $d \geq 0$, and write the fraction on the right-hand side as a telescoping product:

$$\frac{\overline{P}_{S\setminus\Gamma^+(a)}}{\overline{P}_{S\setminus\{a\}}} = \frac{\overline{P}_{S\setminus\{a,b_1\}}}{\overline{P}_{S\setminus\{a\}}} \frac{\overline{P}_{S\setminus\{a,b_1,b_2\}}}{\overline{P}_{S\setminus\{a,b_1\}}} \cdots \frac{\overline{P}_{S\setminus\{a,b_1,\dots,b_d\}}}{\overline{P}_{S\setminus\{a,b_1,\dots,b_{d-1}\}}},$$

where all terms on the right-hand side are strictly positive by the induction hypothesis. By the same hypothesis, each ratio on the right-hand side is bounded above by $\frac{1}{1 - x_{b_i}}$. Therefore,

$$\frac{\overline{P}_{S\setminus\Gamma^+(a)}}{\overline{P}_{S\setminus\{a\}}} \leq \frac{1}{1 - x_{b_1}} \cdots \frac{1}{1 - x_{b_d}}.$$

Recalling that by assumption $\mathbb{P}[\mathcal{E}_a] \leq x_a \prod_{b\in\Gamma(a)}(1 - x_b)$, it follows that

$$\frac{\overline{P}_S}{\overline{P}_{S\setminus\{a\}}} \geq 1 - x_a \prod_{b\in\Gamma(a)} (1 - x_b) \prod_{c\in\Gamma(a)\cap S} \frac{1}{1 - x_c} \geq 1 - x_a > 0.$$

This shows (2.3), and with it, the Lovász Local Lemma. $\square$

References (宋国安硕士论文)

~~Bibliography~~

[1] Noga Alon, Joel H. Spencer, The Probabilistic Method, 2nd ed., *John Wiley and Sons, Inc.*, (2000).

[2] H. B. Chen, D. Z. Du and F. K. Hwang, An unexpected meeting of four seemingly unrealated problems: graph testing, DNA complex screening, superimposed codes and secure key distribution, *J. Combin. Opt.*, to appear.

[3] H. B. Chen, H. L. Fu and F. K. Hwang, An upper bound of the number of tests in pooling designs for the error-tolerant complex model, Opt. Letters, to appear.

[4] P. Damaschke, Threshold Group Testing, *Electronic Notes in Discrete Mathematics 21*, (2005), 265-271.

[5] A. De Bonis, Leszek Gąsieniec, U. Vaccaro, Optimal Two-Stage Algorithms for Group Testing Problems, *SIAM J. Comput. Vol. 34, No. 5*, (2005), 1253-1270.

[6] A. De Bonis and U. Vaccaro, Improved algorithms for group testing with inhibitors, *Inform. Process Lett. 67*, (1998), 57-64.

[7] D. Deng, D. R. Stinson, R. Wei, The Lovász Local Lemma and Its Applications to some Combinatorial Arrays, *Designs, Codes and Cryptography, 32*, (2004), 121-134.

[8] Ding-Zhu Du, Frank K. Hwang, Combinatorial Group Testing and Its Applications, 2nd ed., *World Scientific*, (2000).

[9] Ding-Zhu Du, Frank K. Hwang, Pooling Designs and Nonadaptive Group Testing - Important Tools for DNA Sequencing, *World Scientific*, (2006).

[10] P. Erdős, L. Lovász, Problems and Results on 3-chromatic Hypergraphs and Some Related Questions, *in: Infinite and Finite Sets*(A. Hajnal et al., eds.), North-Holland, Amsterdam, (1975), 609-628.

[11] M. Farach, S. Kannan, E. Knill, S. Muthukrishnan, Group Testing Problem with Sequences in Experimental Molecular Biology, *Proc. Compression and Complexity of Sequences*, (1997), 357-367.

[12] C. H. Li, A sequential method for screening experimental variables, *J. Amer. Statist. Assoc. 57*, (1962), 455-477.

[13] L. Lovász, On the Ratio of Optimal Integral and Fractional Covers, *Discrete Math., 13*, (1975), 383-390.

[14] F. P. Ramsey, On A Problem of Formal Logic, *Proc. London Math. Soc. 30(2)*, (1929), 264-286.

[15] Joel Spencer, Probabilistic Methods, *Graphs and Combinatorics 1*, (1985), 357-382.

[16] D. R. Stinson and R. Wei, Generalized cover-free families, *Discrete Math. 279*, (2004), 463-477.

[17] Hong-Gwa Yeh, d-Disjunct matrices: bounds and Lovász Local Lemma, *Discrete Mathematics 253*, (2002), 97-107.