Introduction to Combinatorics Lecture 7

Edge-coloring

Definition 7.1 (k-edge-coloring)**.** A k-edge-coloring is a mapping $\pi : E(G) \to \{1, 2, ..., k\}$ such that incident edges receive distinct images (colors).

Definition 7.2 (Chromatic index)**.** Chromatic index of G $\chi'(G) = \min\{k \mid G$ has a k-edge-coloring $\}$. If $\chi'(G) = k$, then G is h-edge-colorable for each $h \geq k$.

Theorem 7.1 (Vizing, 1964)**.** *If G is a simple graph, then* $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Proof. The left hand inequality is easy to see. We prove the right hand inequality by induction on $||G||$. We shall prove that G has a $(\Delta(G) + 1)$-edge-coloring (coloring in short) for G and the assertion is true for smaller sizes, i.e., for each $e \in E(G)$, $G - e$ has a coloring π.

First, we observe that since each vertex v is of degree at most $\Delta(G)$, a color is missing around v. Second, if α and β are two colors used in the coloring, then α and β induce a subgraph with components either paths or even cycles. Finally, if 'G has no coloring using $\Delta(G) + 1$ colors', then for each edge xy and any coloring of $G - xy$, there exists an $\alpha - \beta$ path from y ends in x provided α is missing at x and β is missing at y. See Figure 7.1 for missing colors.

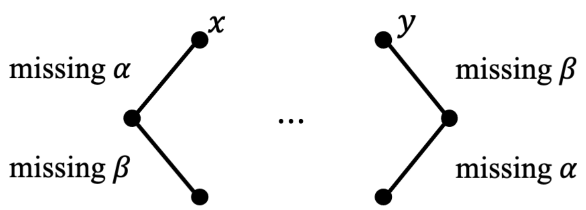


Figure 7.1

Note that if $\alpha - \beta$ path does not connect x and y, then we may recolor one of the path (α, β) to obtain a coloring of G using $\Delta(G) + 1$ colors. Also, if x and y are missing the same color, then we can use that color to color xy and obtain a $\Delta(G)+1$ coloring of G. Hence, it suffices to claim that there is a way to recolor some edges in $G-xy$ such that x and y miss the same color.

Proof of claim. (Outline.)

Let $M(y)$ denote the colors missing at y, and $c_1 \in M(y)$. Now, consider $M(x)$. If $c_1 \in M(x)$, then color xy by c_1 results in a $\Delta(G)+1$ coloring of G. (The claim holds.) Hence, assume $c_1 \notin M(x)$. Let $c_0 \in M(x)$ and $\pi(xy_1) = c_1$, see Figure 7.2. Then, consider $M(y_1)$ and let $c_2 \in M(y_1)$. If $c_2 \in M(x)$, then we let $\pi(xy_1) = c_2$. Thus, c_1 becomes a missing color in $M(x)$, the coloring c_1 is available for xy, $\pi(xy) = c_1$. Hence, assume $c_2 \notin M(x)$. This fact will continue: $c_2 \notin M(x) \Rightarrow \exists y_2$ such that $\pi(xy_2) = c_2$; and then $c_3 \in M(y_2)$, $\pi(xy_3) = c_3; ...;$ $c_{i+1} \in M(y_i)$, $\pi(xy_{i+1}) = c_{i+1}$. Since we only have $\Delta(G)+1$ colors, there exists an l such that $\pi(xy_{l+1}) = c_{l+1} \in \{c_1, c_2, ..., c_l\}$. W.L.O.G., let $c_{l+1} = c_k,\ k \in \{1, 2, ..., l\}$. Now, we have several cases to consider depending on whether $c_0 \in M(y_l)$ or $c_0 \notin M(y_l)$.

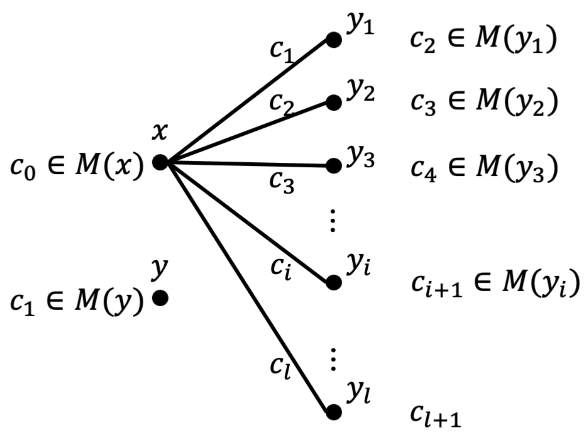


Figure 7.2

Case 1. $c_0 \notin M(y_l)$.

Since $c_{l+1} = c_k,\ c_k \in M(y_l)$. Now, consider $c_k - c_0$ path starting from y_l.

(i) It is a $y_l - y_k$ path. Since $\pi(xy_k) = c_k$, we may recolor them to a $c_0 - c_k$ path starting from y_k. (Note that c_0 occurs in an edge incident to y_l here. By the fact that the last color is c_k, both c_0 and c_k occur an even number of times.) Now, since $\pi(xy_k) = c_0$, the recoloring of $xy_1, xy_2, ..., xy_{k-1}$ gives $c_1 \in M(x)$, we have the proof.

(ii) It is a $y_l - y_{k-1}$ path. Since $c_k \in M(y_{k-1})$, this path is ended with color c_0. That is to say c_0 is also available for xy_{k-1} (not only c_{k-1}). Hence, we color xy_{k-1} with c_0 instead of c_{k-1}, the proof follows by a similar recoloring process as above.

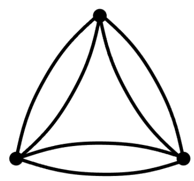
(iii) It is a $y_l - y_i$ path, $i \notin \{k-1, k\}$. Then either c_l or c_0 will be available for xy_i and the proof follows by recoloring process.

Case 2. $c_0 \in M(y_l)$ can be done similarly. □

Base on the same proof technique, we also have a stronger result of Vizing's theorem.

Theorem 7.2 (Vizing, 1964)**.** *If G is a multigraph with multiplicity η, then $\chi'(G) \leq \Delta(G) + \eta$.*

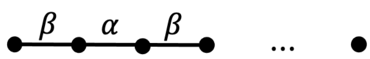
Example. The following graph has $\Delta(G) = 4$ and $\eta = 2$.



Definition 7.3 (Class 1 and Class 2)**.** A graph (simple) is of Class 1 if $\chi'(G) = \Delta(G)$ and of Class 2 if $\chi'(G) = \Delta(G) + 1$.

Theorem 7.3 (König, 1916)**.** *A bipartite graph is of Class 1.*

Proof 1. By induction on $||G||$. Let $xy \in E(G)$ and $G - xy$ can be edge-colored with $\Delta(G)$ colors. Now, since $deg_{G-xy}(x) < \Delta(G)$ and $deg_{G-xy}(y) < \Delta(G)$, a color is missing at x and also a color is missing at y. Let them be α and β respectively. Clearly, $\alpha \neq \beta$, and β occurs around x and α occurs around y. Now, we adapt the idea in proving Vizing's theorem. Let P be a longest $\alpha - \beta$ path from x:



First, if P is an $x - y$ path and the last edge has color α, then P is a path of even length. Hence, $P \cup \{xy\}$ is an odd cycle. A contradiction to the fact that G is bipartite. Hence,

x and y are in different components induced by the set of edges colored α and β. Now, we recolor all the edges of P by interchanging α and β. This gives a coloring in which β is missing at x and also at y. By coloring xy with β, we obtain a $\Delta(G)$-edge coloring of G. □

Proof 2. Let G be a bipartite graph. Then there exists a $\Delta(G)$-regular bipartite graph $\tilde{G} \geq G$. (Exercise) Since $\tilde{G}$ is a $\Delta(G)$-regular bipartite graph, $\tilde{G}$ can be decomposed into $\Delta(G)$ perfect matchings by König's theorem. This implies that $\chi'(\tilde{G}) = \Delta(G)$. Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) = \Delta(G)$. Hence, we conclude the proof. □

Theorem 7.4. *Petersen graph is of Class 2.*

Proof. If G is the Petersen graph and $\chi'(G) = 3$, then G can be decomposed into three 1-factors: F_1, F_2 and F_3 (three color classes). Now, consider the set of five link-edges e_1, e_2, e_3, e_4 and e_5, see Figure 7.3.

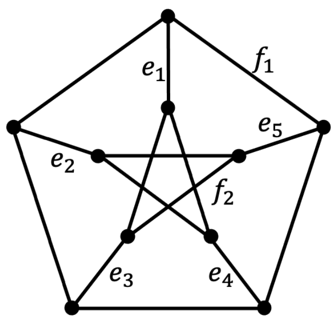


Figure 7.3: Petersen graph.

At least one of F_1, F_2 and F_3 will contain at least two link-edges by Pigeon-hole principle, let it be F_1. Clearly, F_1 cannot contain all the five link-edges. For otherwise, two C_5's is the union of F_2 and F_3 which is impossible. So, there are three cases to consider.

Case 1. $|F_1 \cap \{e_1, e_2, ..., e_5\}| = 4$.

Let e_1 be the edge not in F_1. But, now all the edges of $G - e_1$ not in $\{e_2, e_3, e_4, e_5\}$ are incident to an edge of $\{e_2, e_3, e_4, e_5\}$. So, no other edge can be chosen for F_1.

Case 2. $|F_1 \cap \{e_1, e_2, ..., e_5\}| = 3$.

Let e_1 and e_2 be the edges not in F_1. Then, other than link-edges, we choose at most one more edge f_1. The case e_1 and e_3 are not in F_1 has similar conclusion (only f_2 is available).

Case 3. $|F_1 \cap \{e_1, e_2, ..., e_5\}| = 2$.

This case comes out that we can find two more edges which are not link-edges. □

Corollary 7.5. *Petersen graph contains no Hamilton cycles.*

Proof. If G contains a Hamilton cycle C, then $\chi'(G) = 3$ by coloring the cycle with two colors and $G - C$ (1-factor) with another color. □

Theorem 7.6. *A 3-regular planar graph G is of Class 1.*

Proof. Let G be embedded in S_0. Then, by 4-color Theorem, G is 4-face-colorable (or 4-map-colorable). Let the 4 colors used be obtained from the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$. Since each edge is in the boundary of two adjacent faces, let the edge be colored by $(a_1, b_1) \oplus (a_2, b_2)$ where (a_1, b_1) and (a_2, b_2) are the colors of these two adjacent faces. As a conclusion, we obtain a 3-edge-coloring of G, since $(0, 0)$ will not be used. The coloring is proper since three adjacent faces will receive three different colors, see Figure 7.4. □

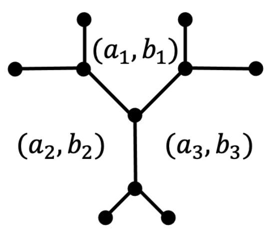


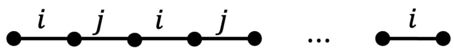
Figure 7.4

Remark. Without using 4CT, the proof is very difficult.

Conjecture 7.1. *If G is planar and $\Delta(G)$ is large enough, then G is of Class 1.*

Theorem 7.7 (Equitable edge-coloring)**.** *If G has a k-edge-coloring f, then G has an equitable edge coloring, i.e., for any two $i, j \in \{1, 2, ..., k\}$, $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$.*

Proof. If there exist i and j such that $||f^{-1}(i)| - |f^{-1}(j)|| \geq 2$, then we consider the graph H induced by the set of edges colored i and j. Then, H is a subgraph of G such that each component of H is either a path or an even cycle. Since i occurs more times than j, there exists an $i - j$ path whose end edges are colored i:



Now, by switching the colors on this path, we obtain a new edge coloring of G such that i occurs one less time and j occurs one more. It turns out that we can obtain an k-edge-coloring such that $||f^{-1}(i)| - |f^{-1}(j)|| \leq 1$. As a consequence, we are able to adjust all of them and obtain an equitable k-edge-coloring. □

Remark. This theorem is not difficult to prove, but very useful.

Definition 7.4 (Overfull)**.** A graph G is said to be overfull if $||G|| > \lfloor \frac{|G|}{2} \rfloor \cdot \Delta(G)$.

Remark.

- If G is overfull, then G is of Class 2.
- If G is overfull, then $|G|$ is odd.

Theorem 7.8. *The complete graph K_n is of Class 2 if and only if K_n is overfull or equivalently n is odd.*

Proof. First, we claim that for each $m \geq 1$, K_{2m} is of Class 1. It is suffices to give a $(2m-1)$-edge-coloring of K_{2m}. For convenience, let $V(K_{2m}) = \mathbb{Z}_{2m} = \{0, 1, 2, ..., 2m-1\}$. For each color $i \in \{1, 2, ..., 2m-1\}$, let the set of edges colored i be
$F_i = \{(0, i), (i+1, i-1), (i+2, i-2), ..., (i+m-1, i-m+1)\}$ (mod $2m-1$). See Figure 7.5 for an example of $m = 5$ and $i = 3$.

Since $\Delta(K_{2m}) = 2m - 1$, $\chi'(K_{2m}) = 2m - 1$.

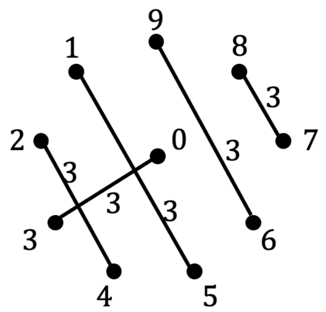


Figure 7.5: $\chi'(K_{10}) = 9$.

Now, by deleting 0 in K_{2m}, we obtain a $(2m-1)$-edge-coloring of K_{2m-1}. On the other hand, it is not difficult to check that K_{2m-1} is overfull for $m \geq 2$, this concludes that $\chi'(K_{2m-1}) > \Delta(K_{2m-1}) = 2m-2$. □

Remark.

- This theorem is not difficult to prove, but it is very useful in the construction of 'Combinatorial Designs'.
- Equivalently, K_{2m} can be decomposed into $2m-1$ 1-factors, which is also known as a 1-factorization of K_{2m}.
- If G is an r-regular graph and $\chi'(G) = r$, then G has a 1-factorization.

Conjecture 7.2. *If G is r-regular and $r \geq \frac{|G|}{2}$, then G has a 1-factorization or equivalently $\chi'(G) = r$.*

Theorem 7.9 (D. Hoffman et al.)**.** *A complete multipartite graph G is of Class 2 if and only if G is overfull.*

Definition 7.5 (Total coloring)**.** A k-total coloring of a graph G is a mapping φ : $V(G) \cup E(G) \to \{1, 2, ..., k\}$ such that

1. adjacent vertices receive distinct images,
2. incident edges receive distinct images, and
3. each vertex has a distinct image with the images of its incident edges.

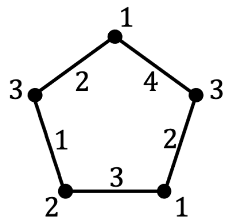


Figure 7.6: A 4-total coloring of C_5.

Definition 7.6 (Total chromatic number)**.** Total chromatic number of G $\chi''(G) = \min\{k \mid G \text{ has a } k\text{-total coloring }\}$.

Theorem 7.10. $\chi''(K_{2n+1}) = \chi''(K_{2n}) = 2n + 1$.

Proof. $\chi''(K_{2n+1})$ can be obtained by using $\chi'(K_{2n+1}) = 2n + 1$. As to the total coloring of K_{2n}, we claim that $2n$ colors are not enough. (Note that $\chi''(G) \geq \Delta(G) + 1$.) Observe that each color class has at most one vertex and $n - 1$ edges. So, $2n$ color classes will contain at most $2n$ vertices and $2n(n - 1)$ edges. Hence, there are $2n^2$ elements (vertices and edges) in total. But, K_{2n} has $2n + \dfrac{2n(2n-1)}{2} = 2n^2 + n$ elements to color,. Clearly, $2n$ color is not enough. Since K_{2n+1} is $(2n + 1)$-total colorable, K_{2n} is also $(2n + 1)$-total colorable. The proof follows. □

Example. $\chi''(K_4) = 5$. (?)

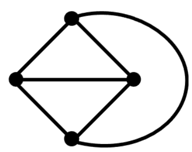


Figure 7.7: K_4

Conjecture 7.3 (TCC Conjecture)**.** $\chi''(G) \leq \Delta(G) + 2$.