Introduction to Combinatorics | Lecture 13

Generating Function

Definition 13.1 (Generating function)**.**

- $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} a_k x^k$.
- $\sum_{k=0}^{\infty} a_k x^k$ is called a generating function of the sequence $\langle a_0, a_1, a_2, ..., a_k, ... \rangle$.

Remark.

- $\binom{n}{k}$ is known as n-choose-k where $n, k \in \mathbb{N} \cap \{0\}$.
- In fact, we can extend n to a real number. In that case, $\binom{r}{k} = r^{\underline{k}}/k!$ where $r^{\underline{k}} = r \cdot (r-1) \cdot (r-2) \cdot \cdots \cdot (r-k+1)$. For example, let $r = \frac{1}{2}$. Then, $\binom{\frac{1}{2}}{5} = \frac{\frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot (-\frac{5}{2}) \cdot (-\frac{7}{2})}{5!}$.
  Also, $(1+x)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k$. (Extension of binomial formula)
  Therefore, we have the geometric series:
  $$(1-x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} (-1)^k x^k = \sum_{k=0}^{\infty} x^k,$$
  since $(-1)^k \binom{-1}{k} (-1) = (-1)^k \frac{(-1)(-2) \cdot (-k)}{k!} = 1$.

Facts

1. $\alpha \sum_{k=0}^{\infty} a_k x^k + \beta \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} (\alpha a_k + \beta b_k) x^k$.
2. (Convolution of two series) $(\sum_{k=0}^{\infty} a_k x^k)(\sum_{k=0}^{\infty} b_k x^k) = \sum_{k=0} (\sum_{i=0}^{k} a_i b_{k-i}) x^k = \sum_{k=0}^{\infty} c_k x^k$,

   i.e., $c_k = \sum_{i=0}^{k} a_i b_{k-i}$.

3. If $F(x) = \sum_{k=0}^{\infty} a_k x^k$, then $F'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$.

Quite a few counting problems can be solved by using G.F., here we present several examples.

Examples

1. How many different ways are there to make a thousand dollars by using Taiwanese coins, 1 dollar, 5 dollars, 10 dollars and 50 dollars.

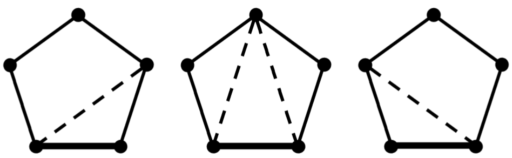
   solution. Let the number of coins be e_1, e_2, e_3 and e_4 respectively for 1, 5, 10 and 50 dollars. Then, $e_1 + 5e_2 + 10e_3 + 50e_4 = 1000$, and the G.F. we can use is

$$(1 + x + x^2 + \cdots)(1 + x^5 + x^{10} + \cdots)(1 + x^{10} + x^{20} + \cdots)(1 + x^{50} + x^{100} + \cdots)$$

$$= \frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{50}}.$$

2. Let h_n denote the number of ways of dividing a convex $(n+1)$-gon into triangles by inserting diagonals which do not cross each other. Find h_n. (Clearly, $h_1 = 1, h_2 = 1, h_3 = 2, h_4 = 5$ and so on.)

   solution. Let $G(x) = \sum_{k=1}^{\infty} h_k x^k$. Observe that $h_n = \sum_{k=1}^{n-1} h_k \cdot h_{n-k}$.



$h_4 = h_1 h_3 + h_2 h_2 + h_3 h_1$

   $[G(x)]^2 = h_1^2 x^2 + (h_1 h_2 + h_2 h_1) x^3 + (h_1 h_3 + h_2 h_2 + h_3 h_1) x^4 + \cdots = G(x) - h_1 x$.

   $[G(x)]^2 - G(x) + x = 0,\ G(x) = \dfrac{1 \pm \sqrt{1-4x}}{2}$.

   Since $G(0) = 0,\ G(x) = \dfrac{1 - \sqrt{1-4x}}{2} = \dfrac{1}{2} - \dfrac{1}{2}(1-4x)^{\frac{1}{2}}$.

   By using Newton's binomial theorem,

$$(1-4x)^{\frac{1}{2}} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n,\ (|x| < \frac{1}{4}).$$

Hence, $G(x) = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}x^n$ and thus $h_n = \frac{1}{n}\binom{2n-2}{n-1}$ $(n \geq 1)$.

- The number $\frac{1}{n}\binom{2n-2}{n-1}$ is known as the Catalan numbers for various n.
- Many counting problems will have their solutions as this number.

3. Following from Example 2, if we would like to partition the $(n+1)$-gon into triangles and one quadrangle, we may use a similar idea to find the number of different ways. (?)

Exponential Generating Functions

Definition 13.2 (Exponential generating function)**.**

- We use the set $\{1, x, x^2, ...\}$ of monomials to define a generating function such as $\sum_{k=0}^{\infty} a_k x^k$.
- If we consider $\langle a_0, a_1, ..., a_n, ...\rangle$ whose terms count permutations, then we shall use monomials $\{1, x, \frac{x^2}{2!}, ..., \frac{x^n}{n!}, ...\}$ to define a generating function: $\sum_{k=0}^{\infty} \frac{a_k}{k!}x^k$.

Examples

- $(1 + x)^n$ is an exponential generating function for $\langle p(n, 0), p(n, 1), ..., p(n, k), ...\rangle$ where $p(n, k)$ denotes the number of k-permutations of an n-element set, in fact $p(n, k)$ is equal to $\binom{n}{k} \cdot k!$:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} \cdot k! \cdot x^k/k!.$$

G.F. E.G.F.

Note that the E.G.F. of sequence $\langle 1, 1, ..., 1, ...\rangle$ is $e^x = \sum_{k=0}^{\infty} x^k/k!$. (This is the reason why we got "exponential".)

- For more examples, please refer to the book "Introductory Combinatorics" by R. A. Brualdi.

Recurrence Relations

- One of the famous sequences is known as the Fibonacci sequence $\langle f_0, f_1, ..., f_n, ...\rangle$ where $f_0 = 0,\ f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.
- In a sequence, if for each n, $a_n = f(a_1, a_2, ..., a_{n-1})$, then we have a recurrence relation f. Clearly, if f is quite complicate, then finding a general form for a_n is also difficult. On the other hand, as mentioned above, in case that the relation is comparatively simple, then there is a hope to settle the sequence and use a close form to represent a_n.

Use G.F. to find f_n

Let $F(x) = \sum_{k=0}^{\infty} f_k x^k$. Since $f_k = f_{k-1} + f_{k-2}$ for $k \geq 2$,

$$\begin{aligned} F(x) &= f_0 + f_1 x + \sum_{k=2}^{\infty} f_k x^k \\ &= x + \sum_{k=2}^{\infty} f_{k-1} x^k + \sum_{k=2}^{\infty} f_{k-2} x^k \\ &= x + x \cdot \sum_{k=2}^{\infty} f_{k-1} x^{k-1} + x^2 \cdot \sum_{k=2}^{\infty} f_{k-2} x^{k-2} \\ &= x + x \cdot (F(x) - f_0) + x^2 \cdot F(x). \end{aligned}$$

$F(x)(1 - x - x^2) = x,\ F(x) = \dfrac{x}{1 - x - x^2} = \dfrac{x}{(1 - \frac{1+\sqrt{5}}{2}x)(1 - \frac{1-\sqrt{5}}{2}x)}$
$= \dfrac{a}{1 - \frac{1+\sqrt{5}}{2}x} + \dfrac{b}{1 - \frac{1-\sqrt{5}}{2}x}$.

Hence, $\begin{cases} a + b = 0 \\ -\dfrac{1+\sqrt{5}}{2} b - \dfrac{1-\sqrt{5}}{2} a = 1 \end{cases}$, $a = \dfrac{1}{\sqrt{5}},\ b = -\dfrac{1}{\sqrt{5}}$.

By geometric series, $f_n = \dfrac{1}{\sqrt{5}}(\dfrac{1+\sqrt{5}}{2})^n - \dfrac{1}{\sqrt{5}}(\dfrac{1-\sqrt{5}}{2})^n$.

Another idea

For the Fibonacci number f_n, we may assume that the solution is of the form q^n for some positive real number q. So, $f_n = f_{n-1} + f_{n-2}$ gives $g^n = g^{n-1} + q^{n-2}$, .i.e., $q^{n-2}(q^2 - q - 1) = 0$.

This yields $q_1 = \dfrac{1+\sqrt{5}}{2}$ and $q_2 = \dfrac{1-\sqrt{5}}{2}$. Since both q_1 and q_2 provide solutions for f_n, so is their linear combination.

The answer is of form $c_1 q_1^n + c_2 q_2^n$ in case that $q_1 \neq q_2$.

Now, we can extend the above idea of a more general linear homogeneous recurrence relation

$$h_n = \sum_{i=1}^{k} a_i h_{n-i},\ \ a_i \neq 0 \text{ is a constant and } n \geq k.$$

- If q is a root of $x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0 \quad (*)$, then $h_n = q^n$ is a solution of the recurrence relation.
- If $(*)$ has k distinct roots $q_1, q_2, ..., q_n$, then $\sum_{i=1}^{k} c_i q_i^n$ is a general solution of h_n and e_i's can be determined by using k initial conditions, $h_0, h_1, ..., h_{k-1}$.

Remark.

- $(*)$ is known as the characteristic equation of the recurrence relation $h_n = \sum_{i=1}^{k} a_i h_{n-i}$.
- If $(*)$ has roots which are multiple, then the situation (solutions) will be different.
- If q is a s-multiple set, then we can check that $h_n = q^n,\ h_n = nq^n, ..., h_n = n^{s-1}q^n$ as solutions, so is the linear combination of them.

Examples

$$h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4},\ h_0 = 1,\ h_1 = 0,\ h_2 = 1 \text{ and } h_3 = 2.$$

Then, $x^4 + x^3 - 3x^2 - 5x - 2 = 0$ has roots $-1, -1, -1$ and 2. So, the general solution for h_n is

$$h_n = c_1(-1)^n + c_2 \cdot n \cdot (-1)^n + c_3 \cdot n^2 \cdot (-1)^n + c_4 \cdot 2^n.$$

By using initial conditions, we obtain

$$h_n = \frac{7}{9}(-1)^n - \frac{1}{3}n(-1)^n + \frac{2}{9}2^n.\ (c_3 = 0)$$

Note that both of the above two conclusions can be proved, again, see Brualdi's book for reference.