Introduction to Combinatorics	Lecture 6

Ramsey Theory

This topic plays an important role in learning the structure of graphs. Moreover, it does have important applications. (?)

Definition 6.1. The Ramsey number $R(s,t)$ is the smallest value "n" for which either a graph G of order n contains K_s or $K_t \leq \bar{G}$ (the complement of G).

Definition 6.2 (Edge-coloring version of Ramsey number)**.** The Ramsey number $R(s,t)$ is the smallest value "n" for which any 2-edge-colored K_n (red and blue), either there exists a red K_s or a blue K_t. (A red K_s is a complete graph of order s such that all its edges are colored red.)

Remark. $R(3,3) = 6$ (Do you know this fact?)

Theorem 6.1. *The following statements are true:*

1. $R(s,2) = s$ *and* $R(2,t) = t$,
2. $R(s,t) = R(t,s)$,
3. *For* $s > 2$ *and* $t > 2$, $R(s,t) \leq R(s,t-1) + R(s-1,t)$, *and*
4. $R(s,t) \leq \binom{s+t-2}{s-1} = \binom{s+t-2}{t-1}$.

Proof.
1. and *2.* are easy to see.

Claim of *3.*
Let $n = R(s,t-1) + R(s-1,t)$. Then, in K_n, each vertex is of degree $R(s,t-1)+R(s-1,t)-1$. Therefore, if K_n is 2-edge-colored by red and blue, then the edges incident to a fixed vertex $x \in V(K_n)$ are either red edges or blue edges. By Pigeon-hole principle, either there are $R(s,t-1)$ blue edges or $R(s-1,t)$ red edges. If the first case holds, then in $\langle N_{K_n}(x) \rangle_{K_n}$ (a complete graph of order $R(s,t-1)$), either there exists a red K_s or a blue K_{t-1}. Hence, we have a red K_s or a blue K_t in K_n. The other case can be obtain by a similar argument.

Claim of *4.*

By inductive argument. (Or induction.)

$$\begin{aligned}
R(s,t) &\le R(s,t-1) + R(s-1,t) \\
&\le \binom{s+t-1-2}{s-1} + \binom{s-1+t-2}{t-1} \\
&\le \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2} \\
&\le \binom{s+t-3+1}{s-1} \\
&\le \binom{s+t-2}{s-1}
\end{aligned}$$

□

Theorem 6.2 (Erdös and Szekeres, 1935)**.** *For each* $s \ge 2$,

$$R(s) \le \frac{2^{2s-2}}{s^{1/2}}.$$

($R(s) =_{def} R(s,s)$*.)*

Proof. $R(s,s) \le \binom{2s-2}{s-1}$. We claim that $\binom{2s-2}{s-1} \le \frac{2^{2s-2}}{s^{1/2}}$ by induction on s.

First, if $s = 2$, $2 \le \frac{4}{\sqrt{2}}$, the assertion is true. Assume that the assertion is true for $s = k$, thus $\binom{2k-2}{k-1} \le \frac{2^{2k-2}}{k^{1/2}}$. Now, we calculate

$$\begin{aligned}
\binom{2k}{k} &= \frac{(2k)!}{k!k!} \\
&= \frac{2k \cdot (2k-1) \cdot (2k-2)!}{k^2 \cdot (k-1)! \cdot (k-1)!} \\
&= \frac{2k(2k-1)}{k^2} \binom{2k-2}{k-1} \\
&\le \frac{4k-2}{k} \cdot \frac{2^{2k-2}}{k^{1/2}} \\
&= \frac{4k-2}{4k} \cdot \frac{2^{2k}}{k^{1/2}}.
\end{aligned}$$

Since $(k+1)^{1/2} \le \frac{4k \cdot k^{1/2}}{4k-2}$, we conclude that $\binom{2k}{k} \le \frac{2^{2k}}{(k+1)^{1/2}}$.

□

Remark.

- The result has been there for almost 50 years before the improvement due to Thomason in 1988: $R(s) \leq \frac{2^{2s}}{s}$.
- The original proof by Ramsey shows that $R(s) \leq 2^{2s-3} = \frac{2^{2s-2}}{2}$. (1930)

Theorem 6.3. *For $k \geq 3$,*

$$R(k) \geq \lceil 2^{k/2} \rceil.$$

Proof. (Probabilistic method)
Consider a random red-blue coloring of the edges of K_n. For a fixed set T of k vertices, let A_T be the event that $\langle T \rangle_{K_n}$ is monochromatic. Hence, $P(A_T) = (\frac{1}{2})^{\binom{k}{2}} \cdot 2$ (red or blue) $= 2^{1-\binom{k}{2}}$. Since there are $\binom{n}{k}$ possible sets for T, the probability that at least one of A_T occurs is $\binom{n}{k} \cdot 2^{1-\binom{k}{2}}$. Now, if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then no event A_T occurs is of positive probability, i.e., there exists a coloring of edges such that no monochromatic K_k occurs. Therefore, for such n, $R(k) > n$.

Let $n = \lfloor 2^{k/2} \rfloor$. It suffices to show that $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$.

$$\begin{aligned}
\binom{n}{k} \cdot 2^{1-\binom{k}{2}} &< \frac{n^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \qquad\qquad (1 - \binom{k}{2} = 1 - \frac{k^2}{2} + \frac{k}{2}) \\
&\leq \frac{(2^{\frac{k}{2}})^k}{k!} \cdot \frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} \\
&\leq \frac{2^{1+\frac{k}{2}}}{k!} \\
&< 1. \qquad\qquad (k \geq 3)
\end{aligned}$$

Hence, $R(k) \geq \lceil 2^{k/2} \rceil$.

□

Remark. Combining Theorems above we obtain: $2^{s/2} \leq R(s) \leq 2^{2s-3}$ for $s \geq 2$.

Open Problem. $R(s) = 2^{(c+o(1))s}$ (c may be equal to 1).

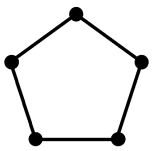
Theorem 6.4. *Known results of $R(s,t)$. ($R(t,s) = R(s,t)$)*

t \ s	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4		18	25	36-41	49-61	59-84	73-115
5			43-48	58-87	80-143	101-216	133-316
6				102-165	115-298	134-495	183-780
7					205-540	217-1031	252-1713
8						282-1870	329-3583
9							565-6588

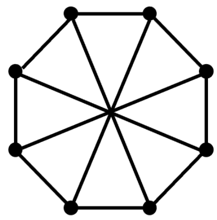
Table 6.1

- The result of lower bounds are obtained by "a special edge-coloring" with two colors. Corresponding to the coloring we have G and $\bar{G}$ of order (prescribed).

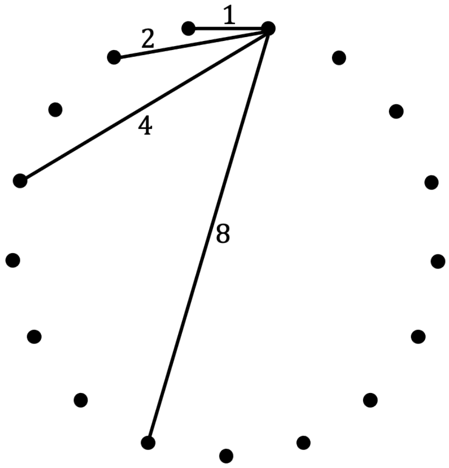
$R(3,3)$: (> 5)



$R(3,4)$: (> 8)



$R(4,4)$: (> 17)



$G(17;\{1,2,4,8\})$

Research Problem.

- Find as many vertices (n) as possible such that a graph G of order n satisfying $K_5 \nleq G$ and $K_5 \nleq \bar{G}$. (Try 43!)
- Find a better upper bound for $R(s)$. (Do your best!)

We can extend the notion $R(s,t)$ to $R(p_1, p_2, ..., p_t)$ by using the coloring version. For $R(s,t)$, we consider 2-coloring the edges of K_n for some n. Now, we color the edges of K_n by using t colors. Hence, we are looking for the existence of monochromatic K_{p_i} using color i (the i-th color).

Definition 6.3. $R(p_1, p_2, ..., p_t) = \min\{n|$ for each t-coloring of $E(K_n)$, there exists a i-monochromatic K_{p_i} for some $1 \leq i \leq t\}$.

Notice that the order of p_i's is important since they may not be the same. In case that $p_1 = p_2 = \cdots = p_t = s$, we denote it by $R_t(s)$. For example, we will prove that
$R_k(3) = \lfloor e \cdot k! \rfloor + 1$. (?) The proof relies on using the generalized Pigeon-hole principle.

Definition 6.4 (Pigeon-hole principle)**.**

- If there are n holes (cages) to hold $n \cdot k - n + 1$ pigeons, then at least one of them will have k pigeons.
- If the n holes are of size $a_1, a_2, ..., a_n$, then $n \cdot k$ can be replaced by $\sum_{i=1}^{n} a_i$ and the i-th hole will have a_i pigeons for some $1 \leq i \leq n$.

Theorem 6.5.

$$R(p_1, p_2, ..., p_t) \leq R(p_1 - 1, p_2, ..., p_t) + R(p_1, p_2 - 1, ..., p_t) + R(p_1, p_2, ..., p_t - 1) - t + 2.$$

Proof. By a similar argument as the proof $R(s,t) \leq R(s, t-1) + R(s-1, t)$.

□

Remark.

- $R(3,3,3) \leq 6 + 6 + 6 - 3 + 2 = 17$ (Theorem 6.6)
- There exists a 3-edge-coloring of K_{16} such that no monochromatic triangles occur.

Theorem 6.6.

$$R(3,3,...,3) =_{def} R_k(3) \leq \lfloor e \cdot k! \rfloor + 1.$$

(k-tuples)

Proof. Since $R(3,3) = 6$, $R(3,3,3) = 17$, the assertion is true for $k = 2$ and 3. Assume that it holds for $k-1$ when $k > 3$. Hence, $R_{k-1}(3) \leq \lfloor e \cdot (k-1)! \rfloor + 1$. By Theorem 6.5,

$$\begin{aligned} R_k(3) &\leq k(\lfloor e \cdot (k-1)! \rfloor + 1) - k + 2 \\ &= k\lfloor e \cdot (k-1)! \rfloor + 2. \end{aligned}$$

Now,

$$\begin{aligned} k\lfloor e \cdot (k-1)! \rfloor &= k\lfloor (k-1)! \cdot (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(k-1)!} + \frac{1}{k!} + \cdots) \rfloor \\ &= k\lfloor M + \frac{1}{k} + \frac{1}{k(k+1)} + \frac{1}{k(k+1)(k+2)} + \cdots \rfloor \\ &\vdots \\ &= \lfloor e \cdot k! \rfloor - 1. \ (?) \end{aligned}$$

□

Remark. Instead of $R(s,t)$, we use $R(H_1, H_2)$ to denote the smallest integer n such that any 2-edge-coloring (red, blue) of K_n, either there exists a red H_1 or a blue H_2.