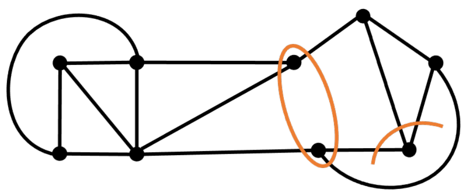
Introduction to Combinatorics Lecture 3

Connectivity

Definition 3.1 (Connectivity)**.** The connectivity of a graph G, $\kappa(G)$, is the minimum number of vertices whose removal from G results in a disconnected graphs or a trivial graph (a graph with one vertex).

Definition 3.2 (Edge connectivity)**.** The edge connectivity of a graph G, $\kappa_1(G)$, is the minimum number of edges whose removal from G results in a disconnected graph.

G:



$\longleftarrow \kappa(G) = 2,\ \kappa_1(G) = 3.$

Theorem 3.1. *For any graph G,*

$$\kappa(G) \leq \kappa_1(G) \leq \delta(G).$$

Proof. Let $v \in V(G)$ and $deg(v) = \delta(G)$. Then, the deletion of all edges incident to v results in a disconnected graph. Hence, $\kappa_1(G) \leq \delta(G)$.

Now, consider the other inequality. First, if $\kappa_1(G) = 0$, then the G is already disconnected, hence $\kappa(G) = 0$. Assume that $\kappa_1(G) > 0$ and let E' be a set of $\kappa_1(G)$ edges such that $G - E'$ is disconnected. Let S be a set of vertices chosen from the set of vertices incident to edges in E' such that each edge is incident to S exactly once. Therefore, $|S| \leq |E'|$. Also, $G - S$ is disconnected or a trivial graph since $G - E'$ is disconnected. This implies that $\kappa(G) \leq |S| \leq |E'| = \kappa_1(G)$. □

Remark.

- G is super-connected if $\kappa(G) = \delta(G)$.
- Let $a \leq b \leq c$ be positive integers. Then, there exists a graph G such that $\kappa(G) = a,\ \kappa_1(G) = b$, and $\delta(G) = c$.

Definition 3.3 (n-connected and n-edge-connected)**.** A graph G is said to be n-connected (resp. n-edge-connected) if $\kappa(G) \geq n$ (resp. $\kappa_1(G) \geq n$).

Remark. A graph is n-edge-connected if it is n-connected.

Definition 3.4 (Separating set)**.** A set S of vertices in G is said to be a separating set of two vertices u and v ((u, v)-separating set) of G of $G - S$ is a disconnected graph in which u and v lie in different components. We also say S separates u and v.

Theorem 3.2 (Manger, 1927)**.** *Let u and v be non-adjacent vertices in G. Then, the minimum number of vertices that separates u and v is equal to the maximum number of internally disjoint $u - v$ paths in G.*

Proof. Many different versions. We include one here for your reference.

Let the number of vertices separating u and v to be k. Then, it is easy to see that there are at most k independent (vertex-disjoint) paths connecting u and v. Also, if $k = 1$, then we have a path joining u and v. Now, suppose the assertion is not true, i.e., we can find less than k independent $u - v$ paths for certain k. Now, take the minimal k in which we have a counterexample. Then, among all such examples, let G be the one with minimum size (number of edge).

First, we notice that u and v have at most $k - 1$ independent paths and no common neighbors. For otherwise, let ux and xv be edges of G. Then $G - x$ will be a counterexample for '$k - 1$' (smaller than k).

Let W be a separating set of u and v and $|W| = k$. Suppose, neither $N_G(u) = W$ nor $N_G(v) = W$. (Figure 3.1)

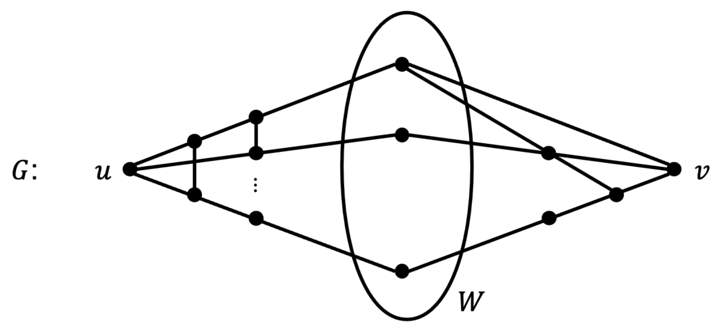


Figure 3.1

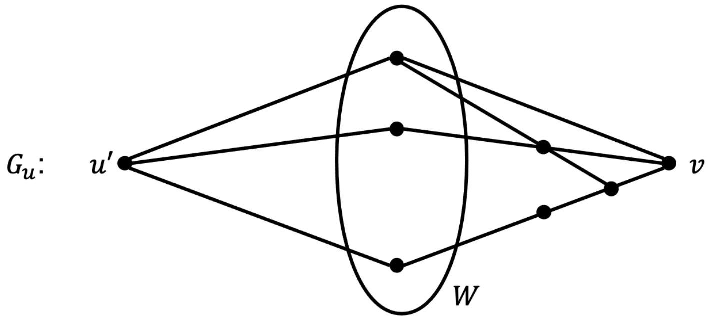


Figure 3.2

Let G_u be obtained by deleting all the vertices to the left of G in Figure 3.1 and adding a replacing u' with edges joining W, see Figure 3.2. Now, G_u has fewer edges than G and thus there are k independent $u' - v$ paths. Hence, we have k $W - v$ independent paths. With the same technique, we derive k $u - W$ independent paths (by changing u to v). So, as a conclusion, either u or v must have their neighbors W. Let $N_G(u) = W$ and $P =< u, x_1, x_2, ..., x_l, v >$ be a shortest $u - v$ path. (Figure 3.3) Then $l \geq 2$. Consider $G - x_1x_2$.

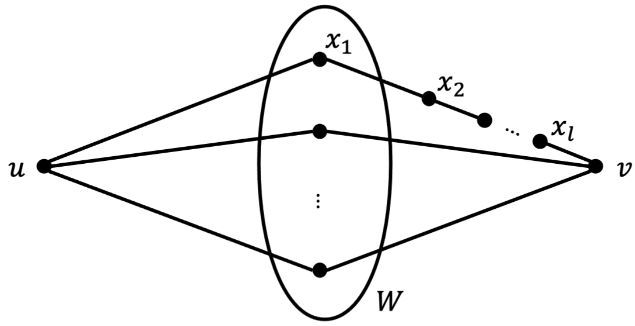


Figure 3.3

In $G - x_1x_2$, there exists a $u - v$ separating set W_0 of size $k - 1$. Then, both $W_1 = W_0$ $\cup\{x_1\}$ and $W_2 = W_0 \cup \{x_2\}$ are $u - v$ separating sets of G. By the fact that P is a shortest $u - v$ path, u is not adjacent to x_2 and v is not adjacent to x_1. This implies that $N_G(u) = W_1$ since v is not adjacent to a vertex of the separating set W_1. Similarly, $N_G(v) = W_2$. Hence, $N_G(u) \cap N_G(v) = W_0$ (u and v have common neighbors), a contradiction. ($|W_0| = k - 1 \geq 1$.) □

Definition 3.5. In G, given a vertex x and a set U of vertices, an $< x, U >$-fan of size k is a set of k internally disjoint (independent) paths from x to U in G.

Theorem 3.3 (Fan Lemma, Dirac, 1960). *A graph is k-connected if and only if it has at least $k + 1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an $< x, U >$-fan of size k.*

Proof.
(⇒) If G is k-connected and $U \subseteq V(G)$ with $|U| \geq k$, then the graph $G' = G + \{yu \mid u \in U\}$ where $y \notin V(G)$ is also k-connected. (?) By Menger's Theorem, there are k internally disjoint paths between x and y in G'. Now, clearly, in G we have an $< x, U >$-fan of size k.

(⇐) It suffices to show that for any two vertices w and z, there are at least k internally disjoint paths. Since an $< x, U >$-fan of size k exists, $deg_G(x) \geq k$, i.e., $\delta(G) \geq k$. Now, let $U = N_G(z)$. By using $< w, U >$-fan, we obtain the desired paths. □

Theorem 3.4. *If G is n-connected ($n \geq 2$) and S is a set of n vertices, then there exists a cycle in G which contains S.*

Proof. By induction on n and clearly the case $n = 2$ is true. Assume that the assertion holds for $n - 1$ and G is an n-connected graph. Now, let $|S| = n$ and $x \in S$. Since G is also $(n - 1)$-connected, $S \setminus \{x\}$ lies on a cycle C (by induction). Furthermore, we have an $< x, V(C) >$-fan of size $n - 1$.

Case 1. $|C| = n - 1$.
The proof follows by finding $\tilde{C}$ which contains all vertices of S, see Figure 3.4.

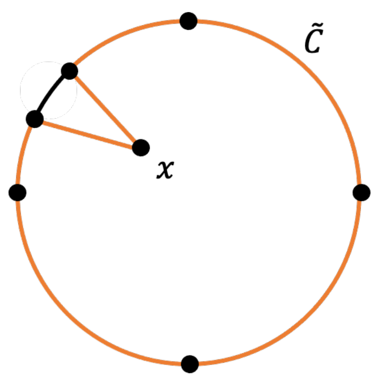


Figure 3.4

Case 2. $|C| > n - 1$.
Since G is n-connected, an $< x, V(C) >$-fan of size n exists. By the fact that $S \setminus \{x\} \subseteq V(C)$, C is partitioned into $n-1$ paths $< V_1, V_2, ..., V_{n-1} >$. Therefore, the $< x, V(C) >$-fan of size n will contain (at least) two vertices in one V_i by Pigeon-hole principle. Now, we are able to find a cycle which contains S. (?) This concludes the proof. □