Introduction to Combinatorics Lecture 1

Subgraphs in Simple Graphs

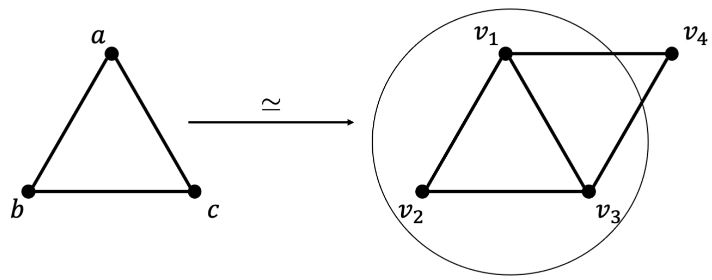
Definition 1.1. A graph G is an ordered pair (V, E) where $V = V(G)$ is the vertex set of G and $E = E(G)$ is the edge set of G.

Definition 1.2. Two vertices u and v in $G(V(G))$ are adjacent, denoted by $u \sim_G v$, if $\{u, v\} = uv$ is an edge of $G(E(G))$ or we say u and v are incident in G. We also say u (and v respectively) is incident to uv.

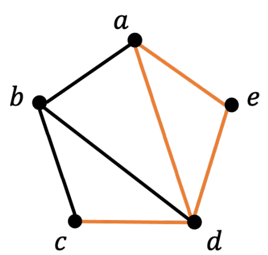
Definition 1.3. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection φ from V_1 to V_2 such that $u \sim_{G_1} v$ if and only if $\varphi(u) \sim_{G_2} \varphi(v)$, denoted by $G_1 \simeq G_2$.

Definition 1.4. A graph $G' = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. (Denoted by $G' \leq G$.)

General sense: $\widetilde{G}$ is a **subgraph** of G if $\widetilde{G}$ is **isomorphic to a subgraph** of G.



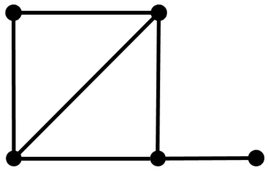
Definition 1.5. (Induced subgraph) Let $S \subseteq V(G)$. Then, the subgraph obtained from S and all edges in G which are incident to two vertices of S is called the induced subgraph of G by S, denoted by $< S >_G$. (Denoted by $< S >_G \preceq G$.)

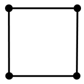


$\longleftarrow$ induced by $S = \{a, c, d, e\}$

Remark. A graph may contain a subgraph H but not an induced subgraph H.

G:



G contains a subgraph C_4 () but not an induced subgraph C_4.

Definition 1.6. The set of vertices in G which are incident to a vertex v is called the neighborhood of v, denoted by $N_G(v)$; and $|N_G(v)|$ is known as the degree of v, denoted by $deg_G(v)$.

Theorem 1.1. *For any graph G, $\sum_{v \in V(G)} deg_G(v)$ is even and the number of vertices with odd degree is also even.*

Proof. Each edge contributes two edges. □

Remark. $\sum_{v \in V(G)} deg_G(v)$ is "known" as the volume of G, which measures how big the graph is.

Definition 1.7. If all degrees of the vertices in G are the same (say k), then G is a regular graph (k-regular). Especially, if k is 3, then we have a cubic graph, and if k is 2, then we have a "2-factor".

Remark.

- The maximum degree (resp. minimum degree) of G is denoted by $\triangle(G)$ (resp. $\delta(G)$). A vertex with the maximum degree is called a major vertex.
- The average degree of G is denoted by $d(G)$.
- $|G|$ is the order of G.
- $\|G\| = |E(G)|$ is the size of G.

Definition 1.8.

- Walk : a sequence of vertices in $V(G)$, $< v_1, v_2, ..., v_m >$, such that for $i = 1, 2, ..., m-1$, $v_i v_{i+1} \in E(G)$.
- Path : a walk with all distinct vertices. (P_m; length $m-1$)

- Cycle : a walk with distinct vertices except $v_1 = v_m$. (C_m; length m)
- Trail : a walk with distinct edges.
- Circuit: a walk with distinct edges and $v_1 = v_m$.

The above definitions are also applied to digraph. ($(v_i, v_{i+1}) \in A(D)$, (v_i, v_{i+1}) is an arc of a digraph D.)

Theorem 1.2. *Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$ provided $\delta(G) \geq 2$.*

Proof. Let $< x_0, x_1, ..., x_k >$ be a longest path we can find in G. Then, $N_G(x_k) \subseteq \{x_0, x_1, ..., x_{k-1}\}$. For otherwise, we have a longer path. Now, $deg_G(x_k) \geq \delta(G)$, but $deg_G(x_k) \leq k$. Hence, $k \geq \delta(G)$ and we have the proof of the first part.

Since $deg_G(x_k) \geq 2$, x_k is incident to some vertex in $\{x_0, x_1, ..., x_{k-2}\}$. Let i be the minimum index in $\{0, 1, 2, ..., k-2\}$ such that $x_k x_i \in E(G)$. Then, $(x_i, x_{i+1}, ..., x_k)$ is a cycle in G. By the fact $deg_G(x_k) \geq \delta(G)$, $i \leq k - \delta(G)$. This implies that the cycle has at least $\delta(G)+1$ vertices. □

Theorem 1.3 (Mantel, 1907)**.** *If $|G| = n$ and $\|G\| > \lfloor \frac{n^2}{4} \rfloor$, then G contains a C_3 (or K_3).*

Proof. Let $x \in V(G)$ be a major vertex, i.e., $deg_G(x) = \triangle(G)$. Assume that $C_3 \nleq G$. This implies that $< N_G(x) >_G$ contains no edges. Hence,

$$\begin{aligned}
\|G\| &\leq \triangle(G) + \triangle(G) \cdot (n - \triangle(G) - 1) \\
&= \triangle(G) \cdot (n - \triangle(G)) \\
&\leq \lfloor \frac{n}{2} \rfloor \cdot (n - \lfloor \frac{n}{2} \rfloor) \\
&= \lfloor \frac{n}{2} \rfloor \cdot \lceil \frac{n}{2} \rceil \\
&= \lfloor \frac{n^2}{4} \rfloor,
\end{aligned}$$

a contradiction. □

Definition 1.9. A graph is called H-free if $H \nleq G$.

Extreme Graph Theory

Research Problem. Given a graph H of order $m \leq n$. Find a graph G of order n which has the maximum number of edges, but G is H-free.

Remark.

- We use $ext(n; H)$ to denote the above mentioned number. The graph which attends this size $ext(n; H)$ is called an extremal graph (which forbids H).
- G is a complete graph of order n if $\|G\| = \binom{n}{2}$, i.e., any two vertices of G are adjacent. We use K_n to denote such graph. $K_{n_1,n_2,\ldots,n_q}$ denotes a **complete multipartite graph** with q partite sets, each of size $n_1, n_2, ..., n_q$ respectively.
- From Theorem 1.3, we have $ext(n; C_3) = \lfloor \frac{n^2}{4} \rfloor$ and $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ is an extremal graph of order n which forbids $C_3 (\simeq K_3)$.

Theorem 1.4 (Turán, 1941)**.** *Let $n = t(p-1) + r$, $1 \leq r \leq p-1$, and*

$$M(n,p) =_{def} \frac{p-2}{2(p-1)} n^2 - \frac{r(p-1-r)}{2(p-1)}.$$

Then, $ext(n; K_p) = M(n,p)$.

Proof. By induction on t. First, if $t = 0$, then $n = r \leq p-1$, clearly, G does not contain K_p. Moreover,

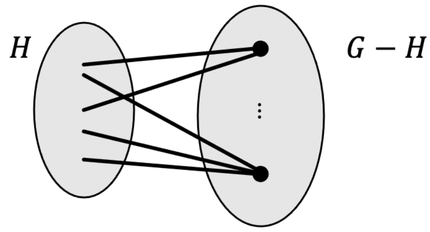
$$M(n,p) = \frac{(p-2)r^2 - rp + r + r^2}{2(p-1)} = \frac{pr^2 - rp - r^2 + r}{2(p-1)} = \frac{(p-1)(r^2-r)}{2(p-1)} = \binom{r}{2},$$

$G \simeq K_r$.

Now, consider $t \geq 1$ and let the assertion be true for $t-1$. Let G be the extremal graph which does not contain K_p. So, G contains K_{p-1}. Let $V(K_{p-1}) = H$. Thus, H contains $p-1$ vertices. Since G does not contain K_p, the vertices outside of H are incident to at most $p-2$ vertices of H. This implies that

$$\|G\| \leq \binom{p-1}{2} + (p-2)(n-p+1) + ext(n-p+1; K_p).$$

Now, $n-p+1 = (t-1)(p-1) + r$. By induction, $ext(n-p+1; K_p) = M(n-p+1, p)$.



$$|H| = p - 1, \quad |G - H| = n - p + 1.$$

Hence,

$$\begin{aligned}\|G\| &\le \binom{p-1}{2} + (p-2)(n-p+1) + \frac{p-2}{2(p-1)}(n-p+1)^2 - \frac{r(p-1-r)}{2(p-1)} \\ &= \frac{p-2}{2(p-1)}\left[(p-1)^2 + 2(p-1)(n-(p-1)) + (n-(p-1))^2\right] - \frac{r(p-1-r)}{2(p-1)} \\ &= \frac{p-2}{2(p-1)}n^2 - \frac{r(p-1-r)}{2(p-1)} \\ &= M(n,p).\end{aligned}$$

For the ($\ge$) direction, let G be the complete multipartite graph $K_{t+1,\dots,t+1,t,\dots,t}$ with r partite sets of size $t+1$ and $p-1-r$ partite sets of size t. Then, $n = r(t+1)+(p-1-r)t = t(p-1)+r$. Now, $\|G\| = M(n,p)$, this concludes the proof.

(G is an extremal graph. In fact, this is the unique extremal graph. (proof?))

□

Definition 1.10. If G contains a cycle, then the length of a shortest cycle is called the *girth* of G, denoted as $g(G)$, and the length of a longest cycle is called the *circumference* of G, denoted as $c(G)$. Clearly, $g(G) \le c(G)$.

Definition 1.11. If $c(G) = |G|$, then G is a hamiltonian graph, i.e., G contains a Hamilton cycle.

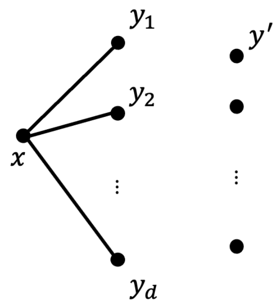
Remark.

- Determining whether a graph is hamiltonian or not is a very difficult problem. But, for the existence of an Eulerian circuit, it is simpler.
- The problem of forbidding cycles of length larger than 3 is comparatively difficult.

Theorem 1.5. *If a graph G of order n has more than $\frac{n\sqrt{n-1}}{2}$ edges, then $g(G) \le 4$. (G contains either a C_3 or a C_4.)*

Proof. Let $g(G) \ge 5$ and $N_G(x) = \{y_1, y_2, ..., y_d\}$. Then, $< N_G(x) >_G$ has no edges (no C_3's).

For vertices $y' \in V(G) \setminus N_G(x)$, y' is incident to at most one vertex in $N_G(x)$ (no C_4's). That is, $N_G(y_i) \cap N_G(y_j) = \{x\}$, for $1 \leq i < j \leq d$. Hence, $\sum_{i=1}^{d} deg_G(y_i) \leq n - (d+1) + d$ $= n - 1$.



Now, consider the volume of G, $vol(G) \leq n(n-1)$.

$$\begin{aligned} n(n-1) &\geq \sum_{x \in V(G)} \sum_{y \sim_G x} deg_G(y) \\ &= \sum_{z \in V(G)} deg_G^2(z) && \text{each } z \text{ of degree } deg_G(z) \text{ will be counted } deg_G(z) \text{ times} \\ &\geq \frac{1}{n}\Big(\sum_{z \in V(G)} deg_G(z)\Big)^2 && \text{by Cauchy's inequality} \\ &= \frac{1}{n}(2\|G\|)^2. \end{aligned}$$

Hence, $\|G\| \leq \frac{1}{2} n \sqrt{n-1}$.

□

Corollary 1.1. *$ext(n; C_3$ or $C_4) \leq \frac{1}{2} n\sqrt{n-1}$. Extremal graphs:*

- $n = 5 : C_5$
- $n = 10 :$ *Petersen graph*
- $n = 50 : srg(50, 7, 0, 1)$ *(strongly regular graph)*

Corollary 1.2. *$ext(n; C_4) \leq \frac{n}{4}(1 + \sqrt{4n-3})$. (proof?)*