Lecture 13

NO. 1

DATE

(t-design)

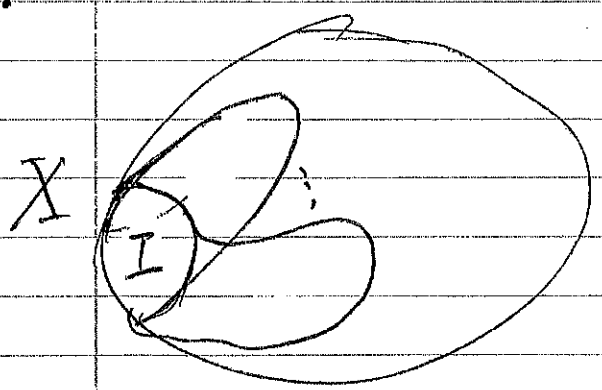
When $t \geq 3$, then finding a good t-design is getting more complicate especially when the block size is also larger. The followings are some basic properties of t-designs.

- If $(X, \mathcal{B})$ is a t-(v,k,λ) design, then

(a) $\lambda \binom{v}{t} / \binom{k}{t}$ is an integer,

(b) for each $0 \leq i \leq t$, the collection of all blocks $\mathcal{B}_i$ containing a given i-subset of X is exactly $\lambda \binom{v-i}{t-i} / \binom{k-i}{t-i}$, and

(c) if I is an i-subset with $i \leq t$, then the collection of blocks $\mathcal{B}_i = \{ B \setminus I \mid B \in \mathcal{B} \}$ is a $(t-i)$-$(v-i, k-i, \lambda)$ design. (with $X_{-i} = X - I$)



SEASON

- Let $k \geq t \geq 2$ Then, the collection of all k-subsets of $X = Z_v$ is in fact a t-design $t-(v,k,\lambda)$ design where $\lambda = \binom{v-t}{k-t}$ if $k > t$ and $\lambda = 1$ if $k = t$.
- If $k=3$ and $t=2$, then $\binom{Z_v}{3}$ forms a $2-(v,3,\lambda)$ design where $\lambda = v-2$. (If $k=4$ and $t=2$, then $\binom{Z_v}{4}$ is a $2-(v,4,\lambda)$ design with $\lambda = \binom{v-2}{2}$.)
- In case of $k=3$, if $\binom{Z_v}{3}$ can be partitioned into $v-2$ disjoint STS(v)'s, then we have a <u>large set of Steiner triple systems.</u>

(*) You may try the case when $v=7$ and $k=3$.

- <u>Theorem</u> (Lu, Jia-Xi, 陸家羲) <u>1935-1983</u> 高中老師

A large set of STS(v)'s exists except for some small cases.

<u>Definition</u> (<u>Steiner systems</u>)

In a t-design, $(X, \mathcal{B})$ if $k = t+1$ and $\lambda = 1$, Then we have a Steiner t-design of order $|X|$. A Steiner triple system of order v is a $2-(v,3,1)$ design and a Steiner quadruple system of order v is a $3-(v,4,1)$ design or $S(t,k,v)$ in short where $t = k-1$.

• Small examples $t=3$ and $k=4$

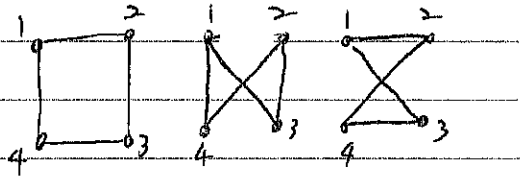
Let $X=\mathbb{Z}_2^4$ and $\mathcal{B}=\left\{\{\vec{w},\vec{x},\vec{y},\vec{z}\}\,\middle|\, \vec{w},\vec{x},\vec{y},\vec{z}\in\mathbb{Z}_2^4,\ \vec{w}+\vec{x}+\vec{y}+\vec{z}=\vec{0}\right\}$.

Notice that $\vec{x},\vec{y},\vec{z}$ and $\vec{w}$ are distinct vectors. Then, $(\mathbb{Z}_2^4,\mathcal{B})$ is an $S(3,4,16)$. It is also true for an $S(3,4,2^m)$ where $m\in\mathbb{N}$.

• Let $X=E(K_5)$ and $\mathcal{B}=\left\{\wedge,\ \triangle,\ \square\ \middle|\ \text{labeled subgraphs of } K_5\right\}$.

$\wedge$: $\binom{5}{1}$, $\triangle$: $\binom{5}{3}$, $\square$: $3\binom{5}{4}$

Then, $(X,\mathcal{B})$ is an $S(3,4,10)$.



$(1,2,3,4)$, $(1,3,2,4)$, $(1,2,4,3)$

• How about $t=3$ and v in general?

• $v\equiv 2$ or $4 \pmod 6$. (Let $(X,\mathcal{B})$ be an $S(3,4,v)$.)

Let $x_0\in X$ and $\mathcal{B}'=\{B\setminus\{x_0\}\mid B\in\mathcal{B}\}$. Then, $(X',\mathcal{B}')$ is an $STS(v-1)$ ($X'=X\setminus\{x_0\}$)

This implies that $|X'|=v-1\equiv 1$ or $3 \pmod 6$.

• Theorem (H. Hanani, 1960)

An $S(3,4,v)$ exists if and only if $v\equiv 2$ or $4 \pmod 6$.

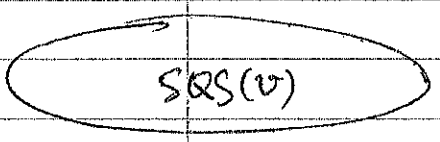
Proof. It takes a lot of effort in proving the sufficient part.

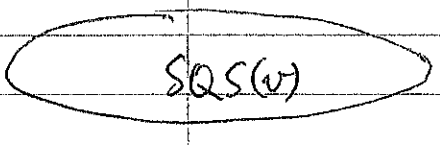
Doubling Construction

Construction I. An $S(3,4,2v)$ exists if an $S(3,4,v)$ exists.

Proof.

(a) (Method 1)





Let K_{X_i} denote the complete graph defined on X_i, $i=1,2$.

Since $|X_i|$ is even, K_{X_i} can be decomposed into 1-factors, there are $v-1$ of them, called $F_1, F_2, \cdots, F_{v-1}$ and $G_1, G_2, \cdots, G_{v-1}$ for $i=1,2$ respectively. Now, we use F_j and G_j, $j=1,2,\cdots,v-1$ to defind $\left(\frac{v}{2}\right)^2$ quadruples by the following way:

$$F_j = \left\{ \{a_i, b_i\} \mid i = 1,2,\cdots,\frac{v}{2} \right\} \Rightarrow \{a_i, b_i, c_j, d_j\} \in \mathcal{B}.$$

$$G_j = \left\{ \{c_j, d_j\} \mid j = 1,2,\cdots,\frac{v}{2} \right\}$$

Combining with $\mathcal{B}_1$ and $\mathcal{B}_2$, we have an $S(3,4,2v)$.

(or $SQS(2v)$).

(*) This $SQS(2v)$ contains two disjoint sub-design $SQS(v)$.

SEASON

NO. 5

DATE

(Method 2)

(b) Let $Y' = \{y' \mid x \in Y\}$ and $X = Y \cup Y'$. Let $(Y, \mathbb{C})$ be an SQS(v).

Define $\mathbb{B}$.

(1) $\forall \{x,y,z,w\} \in \mathbb{C}$, let $\{x,y,z,w\}, \{x,y,z',w'\}, \{x,y',z,w'\}, \{x,y',z',w\}, \{x',y,z,w'\}, \{x',y,z',w\}, \{x',y',z,w\}, \{x',y',z',w'\}$ and $\{x,y',z',w'\}$ be in $\mathbb{B}$.

(2) For any two elements $\{x,y\} \subseteq Y$, let $\{x,y,x',y'\} \in \mathbb{B}$.

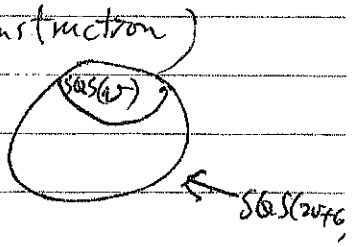
Combining (1), (2), we have an SQS($2v$) $= (X, \mathbb{B})$.

It is a routine matter to check $(X, \mathbb{B})$ is indeed an SQS($2v$) for both constructions: (a) and (b).

The above doubling construction can only handle the cases $v \equiv 4$ or $8 \pmod{12}$. For the other cases, it takes more effort. (We omit the details here.) ($v \to 3v-2$ (Example)

5'

Conjecture $v \to 2v+6$ Construction $(X_1, \mathbb{B}_1)$



For each SQS(v), there exists an SQS($2v+6$) which contains $(X_1, \mathbb{B}_1)$ as a subsystem.

SEASON

Let $q(v) = \frac{v(v-1)(v-2)}{24}$, $p(v) = \frac{v(v-1)}{6}$ and

$q'(v) = q(v) - p(v-1)$.

(•) If $v \equiv 2$ or $4 \pmod 6$, then $p(v-1) = \frac{(v-1)(v-2)}{6}$.

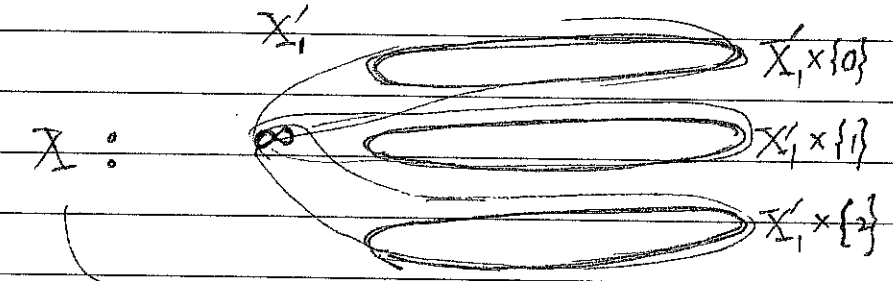
Consider $u \equiv 4$ or $10 \pmod{18}$.

$u = 3v-2$ where $v \equiv 2$ or $4 \pmod 6$. (Example: if $v=8$, then $u=22$.)

Construction of SQS(u)

Let $(X_1, \mathcal{B}_1)$ be an SQS(v) such that $\infty \in X_1$.

Let $X = \{\infty\} \cup \underbrace{(X_1 \setminus \infty)}_{X_1'} \times \mathbb{Z}_3$. So,



Quadruples ($\mathcal{B}$)

1. $\forall \{x,y,z,w\} \subseteq X_1'$ and $\{x,y,z,w\} \in \mathcal{B}_1$, let

$\{(x,a_1),(y,a_2),(z,a_3),(w,a_4)\} \in \mathcal{B}$ where $a_1+a_2+a_3+a_4 \equiv 0 \pmod 3$.

($(a_1,a_2,a_3,a_4) \in \{(0,0,0,0),(0,1,1,1),(1,0,1,1),(1,1,0,1),(1,1,1,0),$

$(0,2,2,2),(2,0,2,2),(2,2,0,2),(2,2,2,0)$, and 18 others$\}$.)

(•) Type 1 quadruples : $\frac{1}{4} \times q'(v)$ quadruples. $\left(\frac{1}{4}\left(\frac{v(v-1)(v-2)}{24} - \frac{(v-1)(v-2)}{6}\right)\right)$

For

2. $\{\infty, u, v, w\} \in \mathbb{B}_1$, define the following quadruples and let them in $\mathbb{B}$:

$\{\infty, (u, b_1), (v, b_2), (w, b_3)\} \in \mathbb{B}$ where $b_1 + b_2 + b_3 \equiv 0 \pmod 3$.

$((b_1, b_2, b_3) \in \{(0,0,0), (1,1,1), (2,2,2), (0,1,2), (0,2,1), (1,0,2), (1,2,0), (2,0,1), (2,1,0)\}.)$

We have $9 \times p(v-1)$ such quadruples. $\left(\frac{9(v-1)(v-2)}{6}\right)$.

3. Continue from 2,

$\{(u,i), (v,i), (w,i+1), (w,i+2)\}$, $i \in \mathbb{Z}_3$.

$\binom{3}{2} \cdot 3$ $\quad$ $9 \cdot p(v-1)$ quadruples.

4. $\forall$ pair (α, β) in X_1', let

$\{(\alpha,i), (\beta,i), (\alpha,i+1), (\beta,i+1)\} \in \mathbb{B}$. $\left(3 \cdot \binom{v-1}{2}\right.$ quadruples$)$.

5. $\forall\ r \in X_1'$, let

$\{\infty, (r,0), (r,1), (r,2)\} \in \mathbb{B}$. $(v-1$ quadruples$)$

In total, we have $\frac{9}{24} \cdot \left(\frac{v(v-1)(v-2)}{24} - \frac{(v-1)(v-2)}{6}\right) + 9 \cdot \frac{(v-1)(v-2)}{6} +$

$9 \cdot \frac{(v-1)(v-2)}{6} + 9 \cdot \frac{(v-1)(v-2)}{6} + (v-1)\Big) = \frac{9}{24} v(v-1)(v-2) + (v-1)$

$= \frac{(3v-2)(3v-3)(3v-4)}{24}$.

(*) If we have $v \to 2v$ and $v \to 2v+6$ constructions, then the theorem about the existence of SQS(v)'s is proved.

Proof. Consider $v \equiv 2$ or 4 or 8 or $10 \pmod{12}$. Clearly, if $v \equiv 4$ or $8 \pmod{12}$, then by $v \to 2v$, we can construct such a system. On the other hand, if $v \equiv 2$ or $10 \pmod{12}$, let $v = 12k+2$ or $12k+10$ respectively. By direct counting, $12k+2 = 2(6k-2)+6$ and $12k+10 = 2(6k+2)+6$. Hence, the construction $v \to 2v+6$ works. ∎

The best known construction besides $v \to 2v$ on SQS(v) is the following

Theorem (Hartman) (Tripling Construction!)

If an SQS(v) contains a subsystem SQS(u), then there exists an SQS($3v-2u$) which contains the above SQS(v).

Note that we can also use this theorem to prove the cases $v \equiv 2$ or $10 \pmod{12}$ (?) except some small cases.

$v \equiv 2$ or $10 \pmod{12}$

$\Rightarrow v \equiv 2, 10, 14, 22, 26, 34 \pmod{36}$

$36k+2 = 3\cdot(12k+2) - 4$ ($u=2$) $S(QS(2)$ is a trivial system

$36k+10 = 3(12k+4) - 2$ $u=1$ (one element)

$36k+14 = 3(12k+10) - 16$ $u=8$

$36k+22 = 3(12k+14) - 20$ $u=10$

$36k+26 = 3(12k+14) - 16$ $u=8$

$36k+34 = 3(12k+14) - 8$ $u=4$

Exercise 4-4 (5 points)

Constructing $SQS(v)$ for as many v as possible.