Lecture 12

On the construction of 2-$(v,4,1)$ designs

Small cases

2-$(13,4,1)$ design $\approx$ projective plane of order 3.

2-$(16,4,1)$ design $\approx$ Affine plane of order 4.

Lemma 1 For each $n \equiv 0$ or $1 \pmod 4$, $n \geq 17$, $n \neq 28$, there exists a pairwise balanced GDD with five (or four) groups G_1, G_2, G_3, G_4, G_5 such that $|G_1| = |G_2| = |G_3| = |G_4| = r$, $|G_5| = r_1 \equiv 0$ or $1 \pmod 4$, $0 \leq r_1 \leq r$, and all blocks are of size <u>4 or 5</u>. Moreover, a $T(r,5,1)$ exists.

Proof. We use $\langle r,r,r,r,r_1 \rangle$ to denote the sizes of groups.

17, 20, 21, ...

$24 \to \langle 5,5,5,5,4 \rangle$ $\qquad 36 \to \langle 8,8,8,8,4 \rangle$ $\qquad 48 \to \langle 11,11,11,11,4 \rangle$

$25 \to \langle 5,5,5,5,5 \rangle$ $\qquad 37 \to \langle 9,9,9,9,1 \rangle$ $\qquad 49 \to \langle 11,11,11,11,5 \rangle$

$29 \to \langle 7,7,7,7,1 \rangle$ $\qquad 40 \to \langle 9,9,9,9,4 \rangle$

[illegible]

$32 \to \langle 7,7,7,7,5 \rangle$ $\qquad 41 \to \langle 9,9,9,9,5 \rangle$ $\qquad n \equiv 0$ or $1 \pmod 4$

$33 \to \langle 8,8,8,8,1 \rangle$ $\qquad 44 \to \langle 9,9,9,9,8 \rangle$

$45 \to \langle 9,9,9,9,9 \rangle$

Per-Duet

For $n \geq 52$, $n = 4r + r_1$ where $0 \leq r_1 \leq r$ and $r_1 \equiv 0$ or $1 \pmod 4$ such a pairwise balanced GDD does exist since there are at least three MOLS(r)'s for each order $r > 10$. ■

Lemma 2 If a 2-$(v,4,1)$ exists, then $v \equiv 1$ or $4 \pmod{12}$.

Proof. It follows by $3 \mid v-1$ and $6 \mid \binom{v}{2}$. ■

Definition (Pairwise balanced GDD, PGDD)

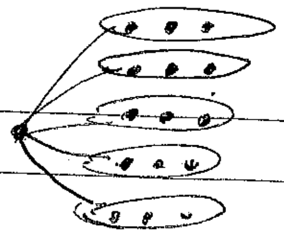
A PGDD $(X, \mathbb{B})$ of order v, denoted by $GD(R, m; S, \lambda)$ is a GDD (general) of order v whose group sizes are in R and block sizes are in S.

e.g. The truncated $T(r,5,1)$ is a $GD(\{r, r_1\}, 5; \{4,5\}, 1)$ of order $4r + r_1$.

Lemma 3

There exists a $GD(3, 5; 4, 1)$.

Proof. Since a 2-$(16,4,1)$ design exists, a $GD(3,5;4,1)$ can be obtained by deleting one element from the design.



Per-Duet

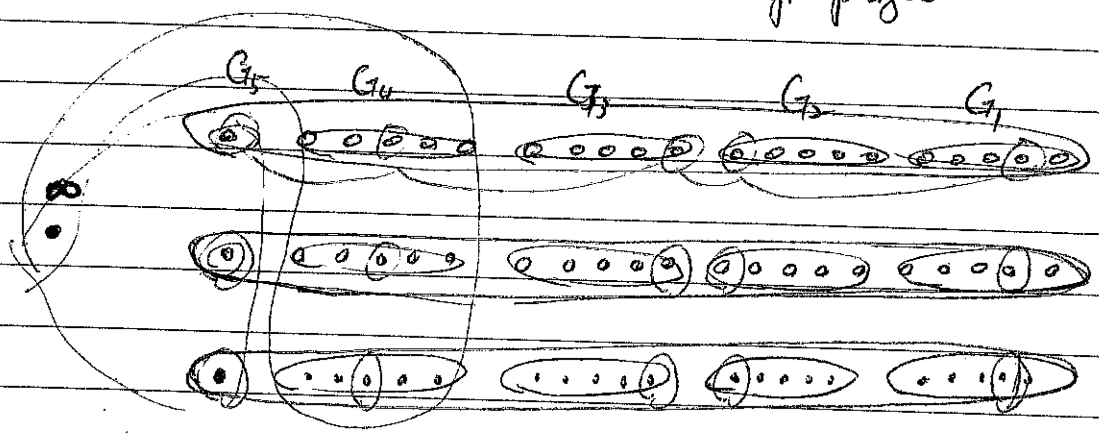
為了容易理解以下就用特殊的例子作說明。

<u>Example</u> A 2-(64,4,1) design exists.

<u>Proof.</u> $64 = 3\times 21+1$

First, we use $r=5$ and $r_1=1$ to obtain a $GD(\{1,5\},5;4,1)$.

\# of groups (5); group sizes ($\{1,5\}$)



<u>Construction</u> $(X, \mathbb{B})$

<u>Step 1.</u> $\mathbb{B}_1$ Use $\{\infty\}\cup(G_i\times\{1,2,3\})$ to construct a 2-(4,4,1) and 2-(13,4,1) respectively.

<u>Step 2.</u> $\mathbb{B}_2$ For each block $B\in GD(\{1,5\},5;4,1)$, construct a $GD(3,4;4,1)$ or $GD(3,5;4,1)$ defined on $B\times\{1,2,3\}$ depends on the size of B, $|B|=4$ or 5. (See the figure above.)

<u>Step 3.</u> Let $X=\{\infty\}\cup\left(\left(\bigcup_{i=1}^{5}G_i\right)\times\{1,2,3\}\right)$, and $\mathbb{B}=\mathbb{B}_1\cup\mathbb{B}_2$.

$|\mathbb{B}_1| = 1 + 4 \times 20 = 81$.

$\Rightarrow |\mathbb{B}| = 336$.

$|\mathbb{B}_2| = 5 \times 15 + 20 \times 9 = 255$

(*) Any two elements occur together in a block. (Check!)

(**) Small orders v s.t. a $2-(v,4,1)$ design exists can be obtained by direct construction. "25" is the hardest one (?

<u>Theorem 4</u> A $2-(v,4,1)$ design exists if and only if $v \equiv 1$ or $4 \pmod{12}$.

<u>Proof.</u> ($\Rightarrow$) By Lemma 1.

($\Leftarrow$) We can construct all $2-(v,4,1)$ designs recursively. Assume that small orders are constructed. Let $v = 3n+1$ where $n \equiv 0$ or $1 \pmod 4$. By the above construction (of example), it suffices to write $n = 4r + r_1$ $(r > 10)$ where $r_1 \equiv 0$ or $1 \pmod 4$ and $3r+1, 3r_1+1 \equiv 1$ or $4 \pmod{12}$.

(For example, if $n = 301$, then we can write $n = 4 \times 72 + 13$ where $r = 72$ and $r_1 = 13$. Since $3 \times 72 + 1 \equiv 1 \pmod{12}$ and $3 \times 13 + 1 \equiv 4 \pmod{12}$, we have a $2-(904, 4, 1)$ design.)

So, we have two cases to consider.

Case 1. $n \equiv 0 \pmod 4$

$n = 12k$, $k > 2$, $r = 12(k-1)$ and $r_1 = 12$.

$n = 12k+4$, $r = 12k$, $r_1 = 4$.

$n = 12k+8$, $r = 12k$, $r_1 = 8$.

Case 2. $n \equiv 1 \pmod 4$

$n = 12k+1$, $k > 2$, $r = 12(k-1)+1$ and $r_1 = 12$.

$n = 12k+5$, $r = 12k$, $r_1 = 5$.

$n = 12k+9$, $r = 12k$, $r_1 = 9$.

This concludes the proof. ∎

How about 2-$(v, k, 1)$ design where $k > 4$?

Observations

1. We need a $GD(R, m; k, 1)$ of certain order n first. $(m \geq k)$

3. $\forall r \in R$, a 2-$(m'r+1, k, 1)$ design exists for each $r \in R$.

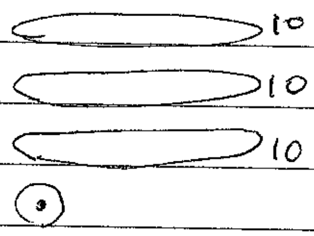
2. A $GD(m', k; k, 1)$ and a $GD(m', k+1; k, 1)$ exists.
($m' = 3$ for $k = 4$.) $\Rightarrow v = 3n+1$.

Review $k=3$ (4^{th} construction!)

Consider $n \equiv 0$ or $1 \pmod 3$.

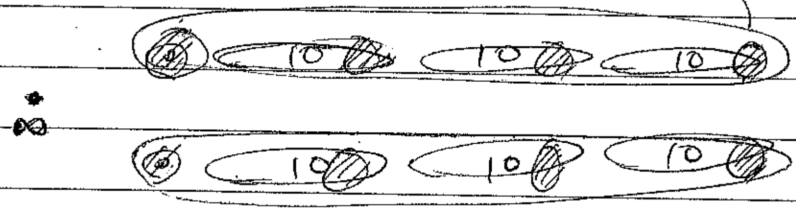
$R = \{3, 4\}$, $n = 3r + r_1$, $r_1 \equiv 0$ or $1 \pmod 3$.

Example $n = 31$



$GD(2,3;3,1)$ and $GD(2,4;3,1)$ exist. $(m' = 2)$

$2-(\geq 1, 3, 1)$ and $2-(3,3,1)$ exist.



Note R中的元素不要選 $x \equiv 2$ or $5 \pmod 6$ 即可。

(*) 當 $\lambda > 1$ 時，建構 $2-(v,k,\lambda)$ 設計也是很重要的課題。
In this case, the set of admissible v is larger that for $\lambda = 1$ in general. For example, if $\lambda = 2$, $k = 3$, then we have: $v \equiv 0$ or $1 \pmod 3$.

$\Downarrow$ 7

Exercise 4-3 (10 points)

Give a detail proof of Theorem 4.

Bonus (5 points)

Say something about $k > 4$.

補充說明

① $K_3 \mid \lambda K_v$ 的充要條件分別為

λ	1	2	3	4	5	6 (mod 6)
v	1, 3	1, 3, 4, 0	1, 3, 5	1, 3, 4, 0	1, 3	0, 1, 2, 3, 4, 5

② $\binom{Z_v}{3} \Rightarrow v-2$ mutually disjoint $STS(v)$'s $\quad (v \equiv 1$ or $3 \pmod 6)$.

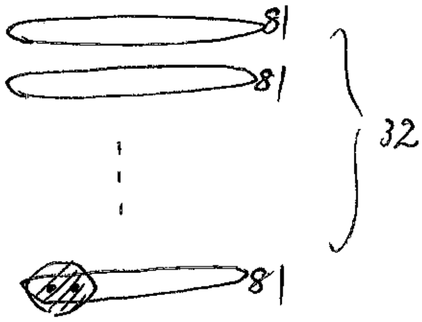
$\Uparrow$

A Large set of triple systems.

(*) An interesting problem to study. Let G be the intersection graph defined on $\mathbb{B}$ where $(X, \mathbb{B})$ is a PBD with index 1 $(\lambda = 1)$. Prove that $\chi(G) \le |X|$. (Especilly consider $(X, \mathbb{B})$ is an $STS(v)$.)

參考用，
作業請用其
它的參數。

A PBD with $\lambda = 1$.



Let $(X, \mathbb{B}) = GD(81, 32; 32, 1) \cong T(81, 32, 1)$.

Truncate two elements from the last group and take groups as blocks. Then, we have a PBD, $2-(2490, \{31, 32, 79, 81\}, 1)$ design.

This implies that there are at least "29" MOLS(2490)'s where $2490 = \underline{2} \times \underline{1245}$.