Lecture 11

Disproof of Euler's Conjecture

Definition (Transversal Designs)

An $(X, \mathbb{B})$, $GDD(n, m; m, \lambda)$, is also known as a transversal design $T(n; m, \lambda)$, i.e., each block is of size m (the number of blocks) or each block $B \in \mathbb{B}$ intersects each group.

(•) Observation

The existence of a pair of orthogonal Latin squares of order n is equivalent to the existence of a $T(n; 4, 1)$ design.

(*) If we can construct a $T(n; m, 1)$, then there exist $m-2$ MOLS(n)'s.

(•) Since a $T(n; n+1, 1)$ exists for prime power n, there exist $n-1$ MOLS(n)'s.

(*) A consequence of $PG(n)$: Delete an element say "0" and then keep all the blocks of size $n+1$.

Another idea of constructing MOLS(n)'s comes from the construction of PBD, $(X, \mathcal{B})$, where $|X| = n$.

<u>Theorem</u> Let $(X, \mathcal{B})$ be $2-(n, K, 1)$ design such that for each $k \in K$, there exist at least t mutually orthogonal Latin squares of order k. Then, there exist $t-1$ MOLS(n)'s.

<u>Proof</u>. If for each $k \in K$, there exists an idempotent Latin square of order k, then there exists an idempotent LS(n) which is obtained from $(X, \mathcal{B})$, see next page for an example.
Therefore, if there are at least $t-1$ idempotent mutually orthogonal Latin squares of order k for each $k \in K$, then we can construct $t-1$ idempotent Latin squares of order n.
(*) (A consequence of the existence of t MOLS(k)'s.)
By the fact that all induced subsquares from blocks are orthogonal, these $t-1$ Latin squares of order n are also <u>mutually orthogonal</u>. ■

(By two finger's rule!?)

Per-Duc

$X = \{0, 1, 2, \cdots, 9\}$

$\mathcal{B} = \{0123, 0456, 0789, 147, 258, 369, 159, 267, 348, 168, 249, 357\}$

L:

0	2	3	1	5	6	4	8	9	7
3	1	0	2	7	9	8	4	6	5
1	3	2	0	9	8	7	6	5	4
2	0	1	3	8	7	9	5	4	6
6	7	9	8	4	0	5	1	3	2
4	9	8	7	6	5	0	3	2	1
5	8	7	9	0	4	6	2	1	3
9	4	6	5	1	3	2	7	0	8
7	6	5	4	3	5	1	9	8	0
8	5	4	6	2	1	3	0	7	9

a	c	b
c	b	a
b	a	c

a	c	d	b
d	b	a	c
b	d	c	a
c	a	b	d

(*) If $\{i,j\}$ occurs in B_k, then use the idempotent Latin square defined on B_k to fill in L_{ij} or $L_{j,i}$ respectively.

Conclusion (i) If $i=j$, then $L_{i,j} = i$.

(ii) If $i \neq j$, L_{ij} is an element of B_k where $\{i,j\} \subseteq B_k$.

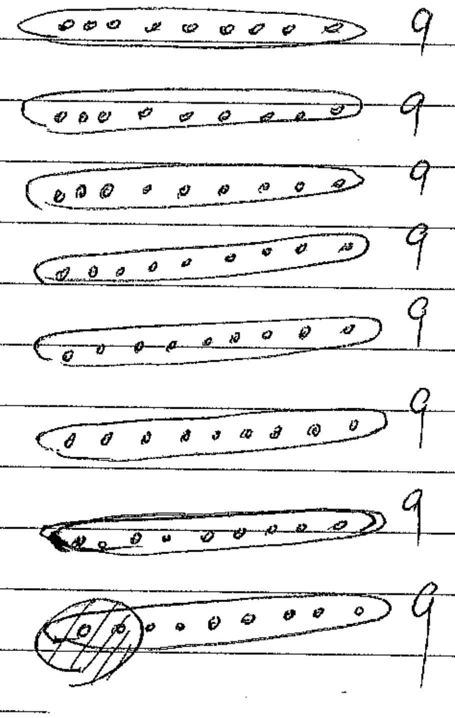
(iii) $L_{i,j} \neq L_{i,j'}$ and $L_{i,j} \neq L_{i',j}$.

($\{i,j\}$ and $\{i,j'\}$, resp. $\{i,j\}$ and $\{i',j\}$ are not in the same block.)

Since a 2-$(22, \{4,7\}, 1)$ design exists, there exists a pair of MOLS(22)'s. In fact, for each $v \geq 19$ and $v \equiv 1 \pmod 3$, a 2-$(v, \{4,7\}, 1)$ design exists. (prove this for a bonus problem.) Hence, a pair of MOLS(v)'s exists for $v \equiv 1 \pmod 3$ and $v \geq 19$

(*) We can use a 2-$(v, \{7,8,9\}, 1)$ design exists for certain v's to show that five MOLS(v)'s exist.

For example, $v = 70$



Start with AG(9) and truncate two elements from the last block.

(5 points)

Exercise 4-1 Find an order $v = 2 \cdot (2m+1)$ such that there exist at least 10 MOLS(v)'s.

Construction of $T(3m+u;4,1)$ where

(1) a $T(m;5,1)$ exists, (2) a $T(u;4,1)$ exists and (3) $m \geq u$.

Step 1 Let $(X_1, \mathbb{B}_1)$ be a $T(m;5,1)$ with five groups G_1, G_2, G_3, G_4, G_5 and $(X_2, \mathbb{B}_2)$ be a $T(u;4,1)$ with four groups $H_i = \{x_1, x_2, x_3, \cdots, x_u\} \times \{i\}$, $i = 1,2,3,4$, i.e., $H_i = \{(x_1,i), (x_2,i), \cdots, (x_u,i)\} =_{def} \{x_{1,i}, x_{2,i}, \cdots, x_{u,i}\}$.

Step 2 Let $(X_1', \mathbb{B}_1')$ be the $T(m;4,1)$ obtained by truncating G_5. Hence, $\mathbb{B}_1'$ can be partitioned into m parallel classes $L_1, L_2, \cdots, L_u, \cdots, L_m$ where each L_j is a set of m disjoint blocks of size 4 defined on $X_1 \setminus G_5 = X_1'$. ($|X_1'| = 4m$, $|\mathbb{B}_1'| = m^2$.)

Step 3 Let X be defined as follows:

$\overline{G}_1$: $G_1 \times \{1\}$, $G_1 \times \{2\}$, $G_1 \times \{3\}$, H_1 ← Each group contains $3m+u$ elements.

$\overline{G}_2$: $G_2 \times \{1\}$, $G_2 \times \{2\}$, $G_2 \times \{3\}$, H_2

$\overline{G}_3$: $G_3 \times \{1\}$, $G_3 \times \{2\}$, $G_3 \times \{3\}$, H_3

$\overline{G}_4$: $G_4 \times \{1\}$, $G_4 \times \{2\}$, $G_4 \times \{3\}$, H_4

An example, $m=7$, $s=5$. (Notation for $T(3m+u;4,1)$)

$G_1 = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ $\quad H_i = \{(x_1, i), (x_2, i), (x_3, i), (x_4, i), (x_5, i)\}$.

$G_2 = \{g_8, g_9, g_{10}, g_{11}, g_{12}, g_{13}, g_{14}\}$

$G_3 = \{g_{15}, g_{16}, g_{17}, g_{18}, g_{19}, g_{20}, g_{21}\}$

$G_4 = \{g_{22}, g_{23}, g_{24}, g_{25}, g_{26}, g_{27}, g_{28}\}$

$\overline{G_1}$: $g_{1,1}\ g_{2,1}\ g_{3,1}\ g_{4,1}\ g_{5,1}\ g_{6,1}\ g_{7,1}\ g_{1,2}\ g_{2,2}\ g_{3,2}\ g_{4,2}\ g_{5,2}\ g_{6,2}\ g_{7,2}\ g_{1,3}\ g_{2,3}\ g_{3,3}\ g_{4,3}\ g_{5,3}\ g_{6,3}\ g_{7,3}$

$x_{1,1}\ x_{2,1}\ x_{3,1}\ x_{4,1}\ x_{5,1}$

$\overline{G_2}$: $g_{8,1}\ g_{9,1}\ g_{10,1}\ g_{11,1}\ g_{12,1}\ g_{13,1}\ g_{14,1}\ g_{8,2}\ g_{9,2}\ g_{10,2} \cdots g_{14,2}\ g_{8,3}\ g_{9,3} \cdots g_{14,3}\ x_{1,2}\ x_{2,2}\ x_{3,2}\ x_{4,2}\ x_{5,2}$

$\overline{G_3}$: $g_{15,1}\ g_{16,1} \cdots g_{21,1}\ g_{15,2}\ g_{16,2} \cdots g_{21,2}\ g_{15,3}\ g_{16,3} \cdots g_{21,3}\ x_{1,3}\ x_{2,3}\ x_{3,3}\ x_{4,3}\ x_{5,3}$

$\overline{G_4}$: . . .

In general, $\overline{G_j} = G_j \times \{1,2,3\} \cup H_j$, $j = 1,2,3,4$.

Therefore, $X = \bigcup_{j=1}^{4} \overline{G_j}$.

The construction follows by defining $\mathbb{B}$ properly!

Step 4 Choose u parallel classes from $(X_1', \mathcal{B}_1')$, say $L_1, L_2, \cdots, L_u$. Starting from $L_1 = \{B_{1,1}, B_{2,1}, \cdots, B_{m,1}\}$, use $(B_{i,1} \times \{1,2,3\}) \cup \{x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}\}$ to construct a $T(4;4,1)$ which contains the block $\{x_{1,1}, x_{2,1}, x_{3,1}, x_{4,1}\}$. → 这个 block 固定，但是 i 是由 1 到 m，因此我们有 m 个 $T(4;4,1)$。一直到 L_u，固定 $\{x_{u,1}, x_{u,2}, x_{u,3}, x_{u,4}\}$。（扣掉固定的不算，一共有 $15mu$ 个 blocks.）

Step 5 For those blocks $B' = \{a,b,c,d\}$ in $L_{u+1} \cup L_{u+2} \cup \cdots \cup L_m$, we use

$$\begin{cases} (a,1),\ (a,2),\ (a,3) \\ (b,1),\ (b,2),\ (b,3) \\ (c,1),\ (c,2),\ (c,3) \\ (d,1),\ (d,2),\ (d,3) \end{cases}$$

to construct a $T(3;4,1)$.

In total, we have $9 \cdot (m-u) \cdot m$ blocks.

Step 6 Construct a $T(u;4,1)$ based on $\bigcup_{j=1}^{u} H_j$.

In total, we have u^2 blocks. Combining Steps 4–6, we have $15mu + 9m^2 - 9um + u^2 = 9m^2 + 6mu + u^2 = (3m+u)^2$.

Step 7 Step 6 shows that the total numbers of pairs we have (from different groups) is $6 \cdot (3m+u)^2$. We have to claim every pair from different groups occurs in a block in Steps 4–6.

Now, consider (y,i) and (z,j), $i \neq j$. If one of them (y and z) is equal to x, then they occurs together in a block either from Step 4 or Step 6. On the other hand, both of them (y and z) are not equal to x, then they occur together in a block of (X_1', B_1'). Hence, they, (y,i) and (z,j), $1 \leq i \neq j \leq 3$, occur together in blocks from Step 4 or 5 depending on whether y and z are in a block of the u parallel classes or not.

(*) Fact: If there exist three MOLS(m)'s and two MOLS(u)'s, then there exists a pair of MOLS(3m+u).

(**) Choose prime powers m and $u \neq 2$ or 6 to construct the desired MOLS(3m+u).

$$
\begin{aligned}
(m,u) \Rightarrow (5,3) &\to 18 \\
(7,1) &\to 22 \\
(7,5) &\to 26 \\
(9,3) &\to 30 \\
(9,7) &\to 34 \\
&\vdots
\end{aligned}
$$

n	m	u
$18k$	$6k-1$	3
$18k+2$	$6k-1$	5
$18k+4$	$6k+1$	1
$18k+6$	$6k+1$	3
$18k+8$	$6k+1$	5
$18k+10$	$6k+1$	7
$18k+12$	$6k+1$	9
$18k+14$	$6k+1$	11
$18k+16$	$6k+5$	1

n, m, u 的選擇

← $k \geq 2$; the case $k=1$ can be replaced by letting $m=9$ and $u=3$.

$(6k+1, 13)$ 在 $k=1$ 時不能用.

Exercise 4-2 (5 points)

Prove that there exists a pair of MOLS(v)'s for each $v \notin \{1, 2, 6\}$.

10.

了 10 階拉丁方陣外，需要更好的組合設計來協助，這一個部分留待第三章再介紹。以下是互相垂直的兩個 10 階拉丁方陣，E.T.Park與 Bose 及 Shrikhande 在 1959 年分別宣稱推翻了尤拉猜測，然而 Bose 及 Shrikhande 兩人的工作似乎獲得較明確的肯定，這應該歸功於他們完整的工作，也就是對於所有的 $n \equiv 2 \pmod 4$，$n \geq 10$，都能建構 2 個 MOLS(n)，而不是只有 10 階拉丁方陣。

4	0	9	8	3	2	7	5	6	1
2	3	7	5	4	0	9	8	1	6
8	1	6	9	0	4	5	3	2	7
9	8	1	4	5	6	3	2	7	0
0	9	8	6	1	3	2	7	4	5
7	2	3	1	6	5	4	0	9	8
5	4	0	3	2	7	6	1	8	9
6	5	4	2	7	1	8	9	0	3
1	6	5	7	8	9	0	4	3	2
3	7	2	0	9	8	1	6	5	4

5	4	0	1	2	7	8	9	3	6
3	1	6	4	8	5	9	2	0	7
0	9	8	7	3	6	1	4	5	2
2	5	4	3	6	1	7	8	9	0
9	8	7	6	1	0	4	5	2	3
1	6	3	5	9	2	0	7	4	8
8	7	2	9	0	4	5	3	6	1
4	0	9	2	7	8	3	6	1	5
7	2	5	0	4	3	6	1	8	9
6	3	1	8	5	9	2	0	7	4

兩個互相垂直的 10 階拉丁方陣

定理 4.11. 對於所有的 n，除了 1,2,6 之外，$M(n) \geq 2$。 ■

證明：$n \equiv 2 \pmod 4$留在第三章證明。

垂直拉丁方陣需要利用組合設計來協助建構，而反過來，我們也可以利用它來建構組合設計，參考第二章。另一方面，在實驗設計上，它也扮演非常重要的角色。

定義 4.12.（**正交表或垂直陣列 Orthogonal Array**）一個深度（Depth）為 k，秩為 n 的正交表 OA(k,n)是一個 $k \times n^2$的陣列，$A=[a_{i,j}]$， 11

12	7	0	6	4	2	11	9	13	5	3	1	10	8
1	11	10	4	5	6	13	8	9	0	7	12	2	3
2	1	11	0	7	12	3	4	5	6	13	8	9	10
10	9	8	1	2	3	4	5	6	13	0	7	12	11
7	0	12	3	1	10	8	6	4	2	11	9	13	5
3	2	1	13	8	9	10	11	0	7	12	4	5	6
5	4	3	10	11	1	2	0	7	12	6	13	8	9
13	6	5	2	3	4	0	7	12	8	9	10	11	1
9	8	13	5	6	0	7	12	10	11	1	2	3	4
11	10	9	8	0	7	12	1	2	3	4	5	6	13
0	12	7	11	9	13	5	3	1	10	8	6	4	2
4	3	2	7	12	5	6	13	8	9	10	11	1	0
6	5	4	12	13	8	9	10	11	1	2	3	0	7
8	13	6	9	10	11	1	2	3	4	5	0	7	12

12	13	4	10	7	0	1	9	6	3	11	8	5	2
9	8	10	5	6	7	4	13	12	11	1	2	3	0
5	0	6	7	8	9	10	11	1	2	3	4	13	12
6	5	7	1	2	3	0	4	13	12	8	9	10	11
4	12	13	6	3	11	8	5	2	10	7	0	1	9
1	11	2	9	10	4	13	12	3	0	5	6	7	8
0	3	5	2	4	13	12	6	7	8	9	10	11	1
7	6	8	4	13	12	9	10	11	1	2	3	0	5
10	9	11	13	12	1	2	3	0	5	6	7	8	4
2	1	3	12	0	5	6	7	8	9	10	11	4	13
13	4	12	3	11	8	5	2	10	7	0	1	9	6
8	7	9	11	1	2	3	0	5	6	4	13	12	10
11	10	1	0	5	6	7	8	9	4	13	12	2	3
3	2	0	8	9	10	11	1	4	13	12	5	6	7

圖 6.3　MOLS(14)