# Lecture 10



## Constructing Designs Using Latin Squares

(• Review)

To start, we use a well-known construction to construct an $STS(v)$ where $v \equiv 3 \pmod 6$. Let $v = 6k+3$ and $L = [l_{ij}]$ be an idempotent commutative Latin square of order $2k+1$. Now, we are ready to construct the Steiner triple system of order $6k+3$.

(1) Let $X = \mathbb{Z}_3 \times \mathbb{Z}_{2k+1}$.

(2) $\forall\, i \in \mathbb{Z}_{2k+1}$, let $\{(0,i),(1,i),(2,i)\} \in \mathbb{B}$.

(3) $\forall\, i<j \in \mathbb{Z}_{2k+1}$, let $\{(0,i),(0,j),(1,l_{ij})\}$, $\{(1,i),(1,j),(2,l_{ij})\}$ and $\{(2,i),(2,j),(0,l_{ij})\}$ be triples in $\mathbb{B}$.

Then, $(X, \mathbb{B})$ is an $STS(6k+3)$.

It is easy to check any two elements of $X$ will occur in a triple and we have in total $(2k+1) + 3 \cdot \frac{(2k+1)^2 - (2k+1)}{2} = 2k+1+6k^2+3k$ $= 6k^2+5k+1 = \frac{(6k+3)(6k+2)}{6}$.

(*) If $(X, \mathbb{B})$ is an $STS(v)$, then $|\mathbb{B}| = \frac{v(v-1)}{6}$.

(**) In difference method, the part $v \equiv 3 \pmod 6$ is comparatively more complicate, we can replace it with this construction if we only try to prove the "sufficient" direction.

We can use MOLS($n$) to construct designs with larger blocks.

(***) The existence of an Affine plane of order $n$ where $n$ is a prim power.

Step 1. Construct $n-1$ MOLS($n$), let them be $L^{(1)}, L^{(2)}, \dots, L^{(n-1)}$.
(For convenience, we use $1, 2, \dots, n$ for $\mathbb{Z}_n$)

Step 2. Let $L^{(r)}$ and $L^{(c)}$ be the row-indices and column-indices squares respectively.

$$L^{(r)} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & & \vdots & \\ n & n & \cdots & n \end{bmatrix} \qquad L^{(c)} = \begin{bmatrix} 1 & 2 & & n \\ 1 & 2 & & n \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 2 & & n \end{bmatrix}$$

Step 3. Let $\overline{X} = (\mathbb{Z}_n \cup \{\infty\}) \times \mathbb{Z}_n = X \cup (\{\infty\} \times \mathbb{Z}_n)$.

Step 4. $\forall\ i, j \in \mathbb{Z}_n$, let $\overline{B_{ij}} = \{(0,i),(1,j),(2,L^{(1)}(i,j)),(3,L^{(2)}(i,j)),\cdots,(\infty,L^{(n-1)}(i,j))\}$ be a block in $\overline{\mathbb{B}}$. (There are $n^2$ blocks.)

Step 5. Let $\mathbb{B}' = \{\overline{B_{ij}} - (\infty, L^{(n-1)}(i,j)) \mid \overline{B_{ij}} \in \overline{\mathbb{B}}\}$.

Step 6. Let $\mathbb{B} = \mathbb{B}' \cup \{\{i\} \times \mathbb{Z}_n \mid i \in \mathbb{Z}_n\}$.

Then, we conclude the $(X, \mathbb{B})$ is an Affine plane of order $n$.

(***) Let $\overline{\overline{X}} = \{(\infty,\infty)\} \cup \overline{X}$ and $\overline{\overline{\mathbb{B}}} = \overline{\mathbb{B}} \cup \{\{(\infty,\infty), \{i\} \times \mathbb{Z}_n\} \mid i \in \mathbb{Z}_n\}$.

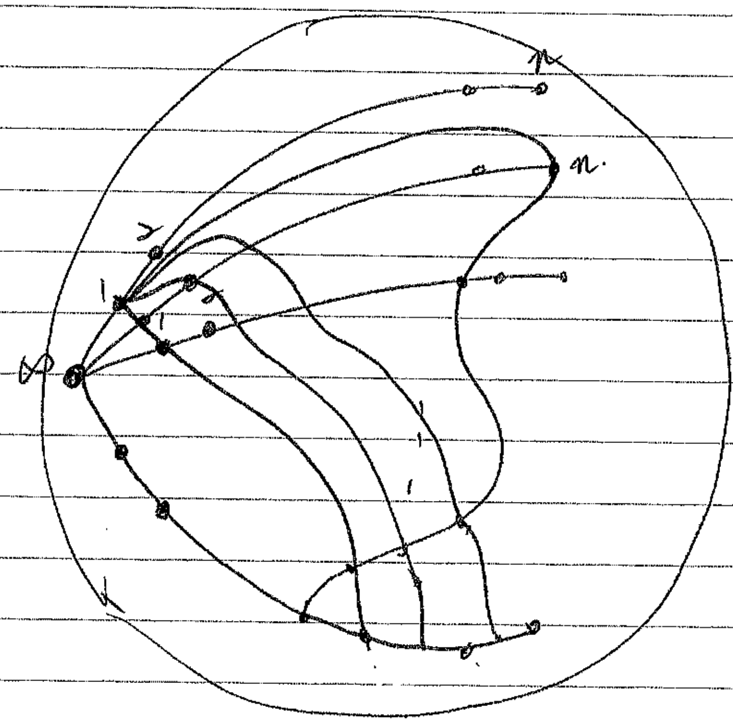
Then $(\overline{\overline{X}}, \overline{\overline{\mathbb{B}}})$ is a projective plane of order $n$.

A complete family of MOLS($n$) $\xleftarrow{?}$ projective plane of order $n$

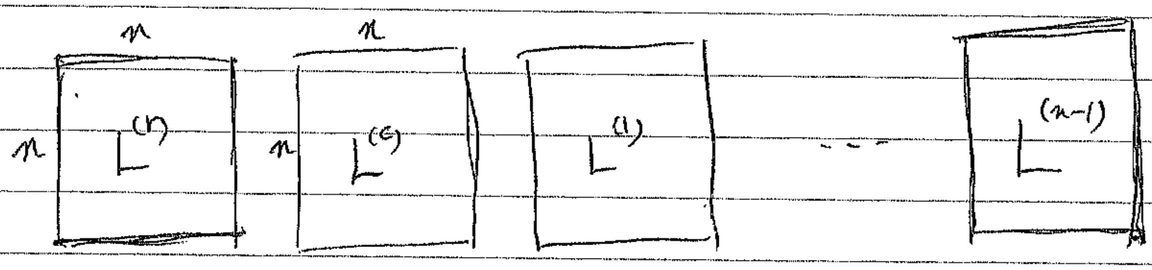
$\approx \Downarrow$

Affine plane of order $n$ $\overset{\approx}{\Longrightarrow}$

$\overset{?}{\Longrightarrow}$




projective plane




$\Downarrow$

3'

We can also use an orthogonal array to construct the desired projective plane of order $n$ where $n$ is a prime power. The steps are similar, except the $k^{th}$ block $B_k$ will be obtained by using the $k^{th}$ column vector of the following array.

$$A: \quad \begin{array}{c} \\ \end{array}\begin{array}{ccc} & B_k & \\ \end{array}$$

$$A: \begin{bmatrix} & & i & & \\ & & j & & \\ & & L^{(1)}_{i,j} & & \\ \cdots & & \vdots & & \cdots \\ & & L^{(n-1)}_{i,j} & & \end{bmatrix}_{(n+1)\times n^2}$$

If the entries of $A$ are in $\{1,2,\cdots,n\}$, then we have $B_k = \{i, n+j, 2n+L^{(1)}_{ij}, \cdots, n^2+L^{(n-1)}_{ij}\}$. So, by adding $\{0,1,2,\cdots,n\}, \{0,n+1,n+2,\cdots,2n\}, \cdots, \{0,n^2+1,n^2+2,\cdots,n^2+n\}$ to the collection of $n^2$ blocks $B_k$'s we obtain the $PG(n)$. Now, an $AG(n)$ can be constructed by deleting $\{0,n^2+1,\cdots,n^2+n\}$ and keep those blocks $B'_k = B_k \setminus \{0,n^2+1,\cdots,n^2+n\}$.

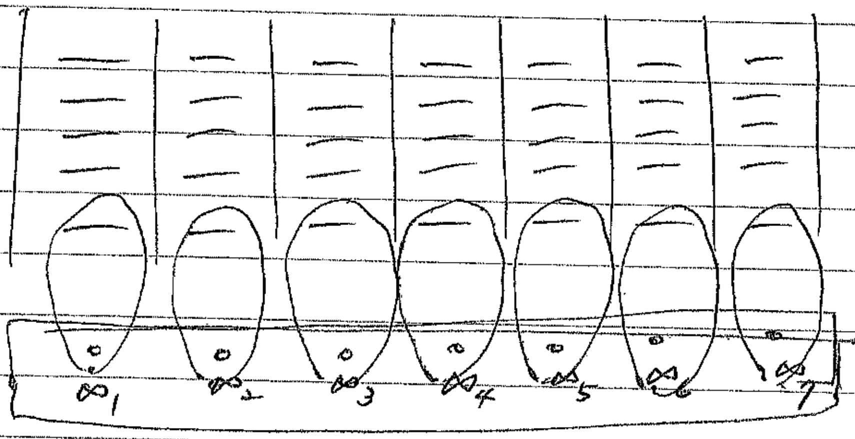
Here, we mention some PBD's.

<u>Theorem 1</u> For each $v \equiv 1 \pmod 3$, there exists a $2$-$(v, K, 1)$-design wh

$K = \{4, 7\}$ except $v = 10, 19$.

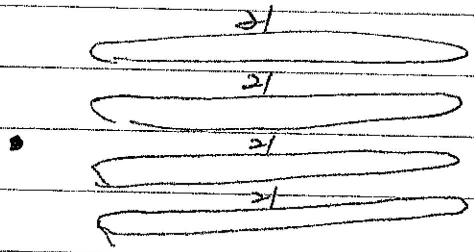
We omit the proof, but we present some examples here.

$v = 22$




By using a Kirkman triple system of order 15, we can attach 7 points in the "infinity" and obtain the desired PBD.

$v = 85$




First, we have a $2$-$(85, \{4, 22\}, 1)$-design by using two MOLS(21). Then, a $2$-$(85, \{4, 7\}, 1)$-design will be obtained from $v = 22$ case.

We can also use MOLS$(n)$ to construct PBDs in which $K$ is of size larger than one. For example, we can use an Affine plane of order 5 to construct a PBD $2-(24,\{4,5\},1)$ design: $(X,\mathcal{B})$. The idea comes from deleting an element from $X$. Then, each block which contains this element becomes a block of size 4, and the other blocks which do not contain this element remain the same.

Hence, we can start with a special type of design, and then either adding or deleting elements (to or from) $X$ to obtain a new design.

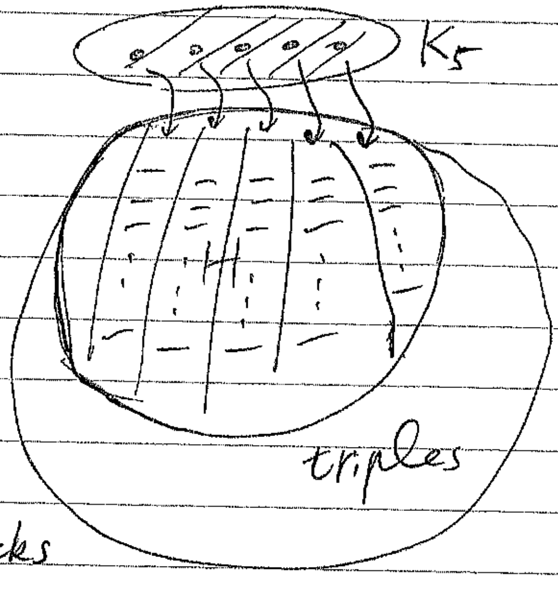
<u>Definition</u> ( Group Divisible Designs of type $n^m$ )

A design $(X,\mathcal{B})$ is called a group divisible design of type $n^m$ if $X$ can be partitioned in $m$ disjoint subsets (called groups) $G_1, G_2, \cdots, G_m$ such that each $B \in \mathcal{B}$, $|B \cap G_i| \le 1$, $|B| = k$ and every pair of two elements from different groups occurs together in exactly $\lambda$ blocks of $\mathcal{B}$. The design $(X,\mathcal{B})$ is denoted by $GDD(n,m;k;\lambda)$.

A GDD$(n,m;k;\lambda)$ can be shorten as a <u>$k$-GDD of type $n^m$ and index $\lambda$</u>. We shall solve the case $k=3$ and $\lambda=1$ in what follows. First, we need a theorem.

<u>Theorem 2.</u> For each $v \equiv 5 \pmod 6$, a $2$-$(v,\{3,5\},1)$-design exists. Moreover, we have such a design with exactly one block of size 5.

<u>Proof.</u> (By difference method.) Let $v=6k+5$ and $X = X_1 \cup X_2$ where $|X_1|=5$ and $|X_2|=6k$. Now, let $X_2 = \mathbb{Z}_{6k}$. Hence, the set of differences in $\mathbb{Z}_{6k} = \{1,2,\cdots,3k\,(\text{half})\}$. As mentioned in the above construction, we can find difference triples either in $\{1,2,\cdots,3k-3\}$ or $\{1,2,\cdots,3k-\cancel{4},3k-2\}$. Hence, after taking away those triples, we have a 5-regular graph $H$ left defined on $\mathbb{Z}_{6k}$. Since $3k$ is one of the differences, $\chi'(H)=5$. The proof then follows by the same idea as in recursive construction.




<u>Note.</u> Such a PBD also exists for $v \equiv 1$ or $3 \pmod 6$ since we can take all blocks of size 3.

Group Divisible Design (3 − GDD)

Problem For which $m$ and $n$, $K_3 \mid K_{m(n)}$?

Fact 1. If $n=1$, then $m \equiv 1$ or $3 \pmod 6$.

Definition (3-sufficient)

A graph $G$ is said to be 3-sufficient if (1) $|G| \geq 3$, (2) $G$ is an even graph and (3) $3 \mid \|G\|$.

Problem (Open) For which 3-sufficient graph $G$, $K_3 \mid G$?

Nash-Williams Conjecture (Remains open)

If $G$ is 3-sufficient and $\delta(G) \geq \frac{3}{4}|G|$, then $K_3 \mid G$.

Fact 2. If $K_{m(n)}$ is 3-sufficient, then

(1) Either $n$ is even or $n$ is odd and $m$ is odd; and

(2) $3 \mid \binom{m}{2} \cdot n^2$.

Theorem If $K_{m(n)}$ is 3-sufficient and $m \geq 3$, then $K_3 \mid K_{m(n)}$

We need several basic facts in order to prove the theor

Fact 3. $K_3 \mid K_{3(n)}$. (By using a L.S. of order $n$.)

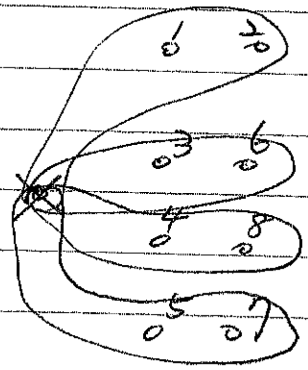
Fact 4. $K_3 \mid K_{4(n)}$ if and only if $n$ is even.

<u>Proof.</u> ($\Rightarrow$)

Since $m=4$, $n$ must be even in order that each vertex is of even degree.

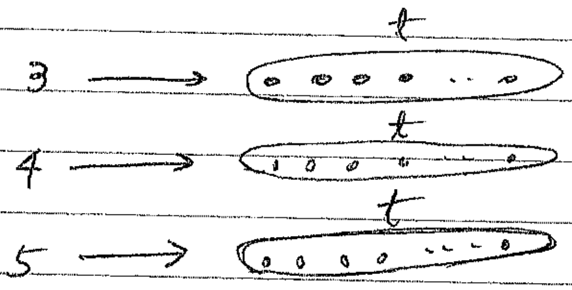
($\Leftarrow$) If $n=2$, then $K_3 \mid K_{4(2)}$. This is a consequence of deleting one vertex of an $STS(9)$. (See it?)

| | | | |
|---|---|---|---|
| 012 | 036 | 048 | 057 |
| 345 | 147 | 156 | 138 |
| 678 | 258 | 237 | 246 |




Now, let $n=2t$. The proof follows by blowing up each vertex into $t$ vertices and use an $LS(t)$ to construct all the $K_3$'s we need.

For example,




As a consequence, we have $8 \cdot t^2$ $K_3$'s in total. This is also the number $K_3$'s we desire: $\frac{6\cdot(2t)^2}{3} = 8t^2$.

<u>Fact 5.</u> If $m \equiv 1$ or $3 \pmod 6$, then $K_3 \mid K_{m(n)}$ for each positive integer $n$.

Proof. It is a direct consequence of blowing each vertex of $K_n$ into $n$ vertices. ■

Fact 6. If $m \equiv 0$ or $4 \pmod 6$ and $n$ is even, then $K_3 \mid K_{m(n)}$.

Proof. First, we take an $STS(2m+1)$ $(X, \mathcal{B})$ and delete one vertex from $X$, then we have $K_3 \mid K_{m(2)}$. Since $n$ is even, we use the same technique as that in Fact 4. The concludes the proof. ■

Fact 7. If $m = 5$ and $3 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. Let $n = 3k$. By the fact that $K_3 \mid K_{5(3)}$, we conclude the proof by blowing each vertex into $k$ vertices. ■

Fact 8. If $m \equiv 5 \pmod 6$ and $3 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. This is a direct result of the existence of a PBD $(m, \{3, k\}, 1)$-design and Fact 7. ■

Fact 9. If $m \equiv 2 \pmod 6$ and $6 \mid n$, then $K_3 \mid K_{m(n)}$.

Proof. Let $m = 6k+2$. Consider $2m+1 \equiv 5 \pmod 6$. Since a $(2m+1, \{3,5\}, 1)$-design exists, we may let it be as in the follow figure.

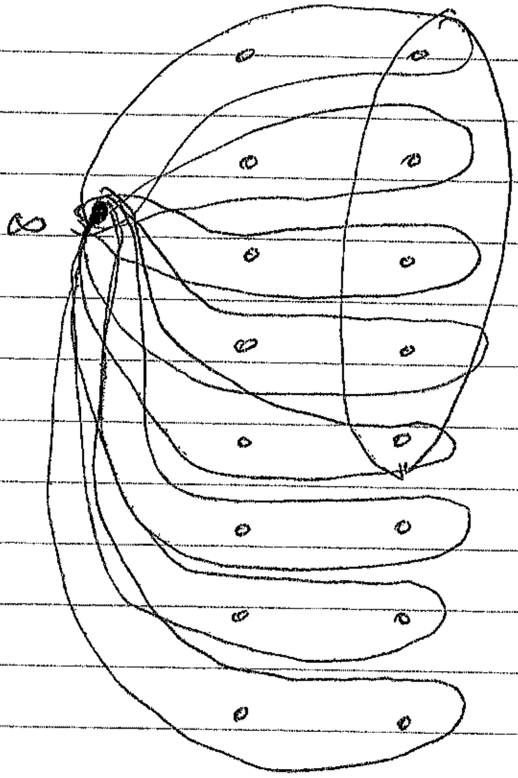



Figure for $(2m+1, \{3,5\}, 1)$ PBD

Now, by deleting $\infty$, we obtain a decomposition of $K_{m(2)}$ into $K_3$'s and one $K_5$. Let $n = 6k$. Then, the proof follows by blowing up each vertex into $3k$ vertices. ■

Theorem (3-GDD)

$K_3 \mid K_{m(n)}$ if and only if $K_{m(n)}$ is 3-sufficient.

Proof. Combining Facts 5, 6, 7, 8, 9; we have the proof. ■

Exercise 3-3. (.5 points) Prove the above theorem in details.