Lecture 9

Recursive Constructions

We may also use the idea of recursion to construct all STS(v). There are two constructions.

1. $v \to 2v+1$ (If an STS(v) exists, then an STS($2v+1$) exists.)

Since $v \equiv 1$ or $3 \pmod 6$, K_{v+1} is a complete graph of even order, and then K_{v+1} can be decomposed into v 1-factors by way of $\chi'(K_{v+1}) = v$. Let $F_1, F_2, \cdots, F_v$ be the set of 1-factors mentioned above. Now, we are ready to construct an STS($2v+1$) $= (\mathbb{Z}_{2v+1}, \mathbb{B})$. Let the given STS($v$) be defined on $\{0, 1, 2, \cdots, v$ and $V(K_{v+1}) = \{v, v+1, \cdots, 2v\}$. Moreover, let $F_i = \{\{a_1^{(i)}, b_1^{(i)}\}, \cdots, \{a_{\frac{v+1}{2}}^{(i)}, b_{\frac{v+1}{2}}^{(i)}\}\}$ be the ith 1-factor, $i = 1, 2, \cdots, v$. So, $\mathbb{B}$ can be obtained by the following:

(a) If B is a triple (block) in STS(v), then $B \in \mathbb{B}$; and

(b) for each $i \in \{0, 1, 2, \cdots, v-1\}$, $\{i, a_j^{(i+1)}, b_j^{(i+1)}\} \in \mathbb{B}$ where $\{a_j^{(i+1)}, b_j^{(i+1)}\} \in F_{i+1}$. (Use $\langle i, F_{i+1} \rangle$ for convenience.)

It is a routine matter to check that $(X, \mathbb{B}) = (\mathbb{Z}_{2v+1}, \mathbb{B})$ is an STS($2v+1$)

2. $v \to 2v+7$

This construction is more complicate comparing to the first one. The main idea comes from the graph $K_{v+7} \approx G(v+7; D)$ where $D = \{1, 2, \cdots, \frac{v+7}{2}\}$. That is, we can view K_{v+7} as a circulant graph with difference set D. By * Stern and Lenz's Lemma, $G =_{def} K_{v+7} \setminus G(v+7, \{1,2,3\})$ can be v-edge-colored for each $v \geq 3$. This implies that G can be decomposed into v 1-factors $F_1, F_2, \cdots, F_v$. Now, we are ready to construct an $STS(2v+7)$ by way of an $STS(v)$ defined on $X = \{0, 1, 2, \cdots, v-1\}$. Let $(X, \mathbb{B}_1)$ be an $STS(v)$, and $STS(2v+7) = (\mathbb{Z}_{2v+7}, \mathbb{B})$.

It suffices to find $\mathbb{B}$. The triples of $\mathbb{B}$ are obtained as follows:

(a) $\forall\ B \in \mathbb{B}_1$, $B \in \mathbb{B}$;

(b) Decompose $G(v+7; \{1,2,3\})$ into K_3's defined on $\{v, v+1, \cdots, 2v+6\}$ and let each of them be a triple of $\mathbb{B}$; and

(c) $\langle i, F_{i+1} \rangle \subseteq \mathbb{B}$ for each $i = 0, 1, \cdots, v-1$. ($\langle i, F_{i+1} \rangle$ is similar to (b) in Case 1.)

Again, it is not difficult to check $(\mathbb{Z}_{2v+7}, \mathcal{B})$ is indeed an $STS(2v+7)$.

Based on the above two constructions, we conclude the proof by showing each $STS(u)$ can be obtained by recursive constructions $v \to 2v+1$ or $v \to 2v+7$. First, if $u = 6t+1$, then $u = 12s+1$ or $12s+7$. Since $12s+1 = \underbrace{(6s-3)}_{\equiv 3 \, (\mathrm{mod}\, 6)} \cdot 2 + 7$ and $12s+7 = (6s+1) \cdot 2 + 1$, an $STS(u)$ can be constructed recursively. On the other hand, if $u = 6t+3$, then $u = 12s+3$ or $12s+9$. Since $12s+3 = (6s+1) \cdot 2 + 1$ and $12s+9 = (6s+1) \cdot 2 + 9$, an $STS(u)$ can be constructed by the same reason. This concludes the proof. $\blacksquare$

Exercise 3-1: (15 points) (再提醒一次!)

Prove that for each $v \equiv 1$ or $3 \pmod 6$, an $STS(v)$ exists by using three distinct constructions. (Not limited to the three ways provided in this note.)

補充說明

<u>Theorem</u> (Stern and Lenz) (Bonus, prove this theorem.)

Let $G(n;D)$ be a circulant graph with difference set D. If $\frac{n}{2}$ is an integer and $\frac{n}{2} \in D$, then $G(n;D)$ is of Class 1.

This theorem can be applied to prove the well-known Doyen-Wilson theorem on Steiner triple systems.

<u>Theorem</u> (Doyen and Wilson, 1973)

An $STS(v)$ can be embedded in an $STS(u)$ if and only if $u \geq 2v+1$.

<u>Proof.</u> ($\Rightarrow$) Let $(X_1, \mathbb{B}_1)$ be an $STS(v)$ and $(X, \mathbb{B})$ be an $STS(u)$ such that $X_1 \subseteq X$ and $\mathbb{B}_1 \subseteq \mathbb{B}$. Now, consider a fixed element in $X \setminus X_1$, say x_0. Then, for each element $x_i \in X_1$, the triple contain x_0 and x_i should be $\{x_0, x_i, y_i\}$ where $y_i \in X \setminus X_1$. Since there are v elements in X_1, $X \setminus X_1$ contains x_0, y_i, $i = 1, 2, \cdots, v$. Hence, $u \geq 2v+1$.

($\Leftarrow$). It takes some effort to finish the proof.

3'' provides some cases.

補充說明

Case 1. $v = 6k+1$ and $u = 6h+3$ where $u \geq 12k+3$, i.e, $h \geq 2k$.

By the idea of recursive constructions, we define a complete graph G of order $\underbrace{u-v}=6(h-k)+2$ defined on $[0,u-1]\setminus[0,v-1]$. Therefore, $G \cong G(u-v; D)$ where $D=\{1,2,\cdots,3(h-k)+1\}$. Now, G can be decomposed into v 1-factors and a collection of triples. See the following example to describe the idea.

Example, $v=13$, $u=45$.

$G =_{def} G(32;\{1,2,\cdots,16\})$. Let $D'=\{9,11,12,13,14,15,16\}$. By Stern and Lenz's lemma, $G(32;D')$ can be partitioned into 13 1-factors. On the other hand $G(32;\{1,2,3,4,5,6,7,8,10\})$ can be partitioned into cyclic triples by using extended Skolem sequence.

Case 2. $v=6k+3$ and $u=6h+3 \geq 12k+7$, i.e., $h \geq 2k+1$.

Example, $v=15$, $u=45$. $G =_{def} G(30,\{1,2,\cdots,15\})$

Now, $D'=\{1,6,8,9,11,12,13,14,15\}$, $D_1=\{10\}$ (short orbit), $D_2=\{1,2,3,4,5,7\}$ (Full orbits). $G(30;D')$ can be partitioned into 15 1-factors and $G(30;D_1 \cup D_2)$ can be decomposed into triples.

補充說明

Case 3 $v = 6k+1$ and $u = 6h+1$.

Example $v = 13$, $u = 43$.

(i) $G(30;\{1,2\})$ can be decomposed into 10 triangles $\mathbb{B}_0$ and 1 Hamilton cycle (two 1-factors), let them be F_1 and F_2.

Let $D' = \{5,8,12,13,14,15\}$. Together with F_1 and F_2 and $G(30;D')$, we have 13 1-factors. For the other differences $D'' = \{3,4,6,7,9,11\} \cup \{10\}$ $G(30;D'')$ can be decomposed into triangles $\mathbb{B}_1$. Combine $\mathbb{B}_0, \mathbb{B}_1$, STS(13) and (a_i, F_i)'s we have the embedding.

Case 4 is similar to Case 3.

Maximum Packing of a graph G with triangles

Definition (MTS(v))

A maximum packing of K_v with triangles, denoted by MTS(v), is a design $(X, \mathbb{B})$ such that $|X| = v$, and $|\mathbb{B}|$ is the maximum number of triples we can find in the set $\mathbb{Z}_v$.

(o) For each $v \equiv 1$ or $3 \pmod 6$, an MTS(v) is in fact an STS(v). So, it's left to consider the other 4 cases.

Definition (Leaves)

If $v \equiv 0, 2, 4, 5 \pmod 6$, then there exists a non-empty subgraph of K_v, say L_v, such that $K_v - L_v$ can be decomposed into triangles. If L_v is of minimum size, then L_v is referred to as a minimum leave.

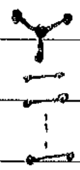
(oo) Similarly, we can consider a general graph. 顯然不含三角形的圖不需要太多的考量。

Per-Duet

<u>Theorem</u> In an $MTS(v)$, L_v is as following table.

L_v:

L_v \ v	0	1	2	3	4	5 (mod 6)
L_v	F	$\emptyset$	F	$\emptyset$	T	C_4

F : 1-factor, T : Tri-pole 

<u>Proof.</u> We have the cases $v \equiv 1$ or $3 \pmod 6$ in previous lectures.

<u>Case 1.</u> $v \equiv 0$ or $2 \pmod 6$.

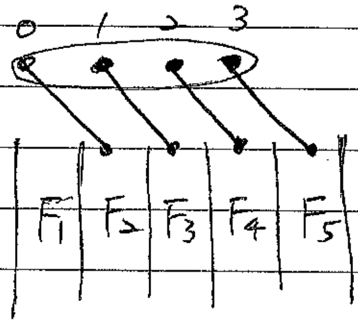
Since each vertex is of odd degree, L_v is a spanning odd subgraph of K_v. By direct calculation, a 1-factor gives the desired leave. In fact, this can be done by deleting a vertex of K_{v+1} ($STS(v+1)$).

<u>Case 2.</u> $v \equiv 5 \pmod 6$

Notice that $K_v - K_5$ can be decomposed into triangles (?). Now, taking away two triangles from K_5, we have the desired leave C_4. (Since L_v in this case is an even subgraph of size $3k+1$, C_4 is the best choice (unique choice).

Case 3. $v \equiv 4 \pmod 6$.

We can see this leave by looking at $v=10$.



So, combining L_4, a star $K_{1,3}$, and F_1, we have the desired leave.

Definition (k-sufficient)

If a graph G satisfying that (1) each vertex is of even degree, (2) $|G| \geq k$ and (3) $k \mid \|G\|$, then G is said to be k-sufficient.

Open problem

For which graphs G that G is k-sufficient and $C_k \mid G$?

(✱) How about $k=3$?