Lecture 8

DATE

BIBD with small block sizes

- A $2-(v,2,\lambda)$ design exists for all $v \geq 2$.

  This is a direct consequence of using $\lambda \cdot K_v$.

- A $2-(v,3,1)$ design exists if and only if $v \equiv 1$ or $3 \pmod 6$

  This theorem was first proved by T. P. Kirkman in 1847.

Later, there are many different proofs for this seemingly easy but quite complicate "fact".

Theorem 1. A $2-(v,3,1)$ design, known as a Steiner triple system of order v, exists if and only if $v \equiv 1$ or $3 \pmod 6$.

Proof.

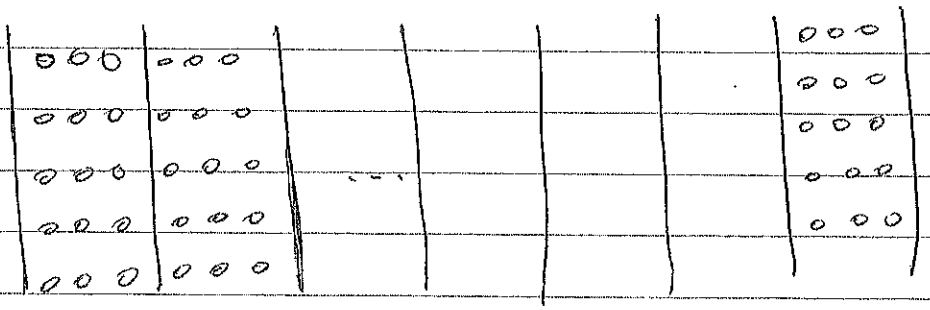
N. C. ($\Rightarrow$) As mentioned earlier, if a $2-(v,3,1)$ design exists then $r = \frac{v-1}{3-1} = (v-1)/2$ and $b = \frac{v(v-1)}{6}$ are both integers. This implies that $v \equiv 1$ or $3 \pmod 6$.

S. C. ($\Leftarrow$) We prove this sufficient condition by constructing a $2-(v,3,1)$ design for each $v \equiv 1$ or $3 \pmod 6$.

NO. 1

DATE

Kirkman's 15 School Girls Problem



Problem 安排15個女學生排成5列每列3人走路上學，可否在七天內完成任兩位同學都在不同/含在同一列？

Answer: We need at least 7 days, since each day we use up 15 pairs and in total there are $\binom{15}{2} = 105$ pairs. So, the extra requirement is that every day, the arrangement is in fact a parallel class. Such designs are also known as Kirkman triple systems of order v. Such systems exist if and only if $v \equiv 3 \pmod 6$.

Note. $AG(2,3)$ is a Kirkman triple system of order 9.

Here is an answer of 15 girls problem.

0 1 2	0 3 4	0 5 6	0 7 8	0 9 10	0 11 12	0 13 14
3 7 11	1 7 9	1 8 10	1 11 14	1 12 13	1 3 5	1 4 6
4 9 13	2 12 14	2 11 13	2 4 5	2 3 6	2 8 9	2 7 10
5 10 12	5 8 13	3 9 14	3 10 13	4 8 11	4 10 14	3 8 12
6 8 14	6 10 11	4 7 12	6 9 12	5 7 14	6 7 13	5 9 11

First, we need to construct Steiner triple systems of small orders, $v = 7, 9, 13$ and 15. (Defined on $\mathbb{Z}_v$.)

$v = 7$ $\quad 013, 124, 235, 346, 450, 561, 602.$ (PG(2))

$v = 9$ $\quad 012, 345, 678, 036, 147, 258,$ (AG(3))
$048, 156, 237, 057, 138, 246.$

$v = 13$ $\quad \mathcal{B} = \{(0,3,4)+i, (0,2,7)+i \pmod{13} \mid i \in \mathbb{Z}_{13}\}$. (PG(3))

$v = 15$ $\quad \mathcal{B} = \{(0,3,4)+i, (0,2,8)+i, (0,5,10)+i \pmod{15} \mid i \in \mathbb{Z}_{15}\}$.

Now, we shall use the following two constructions to construct all the other Steiner triple systems of order v, STS(v) in short.

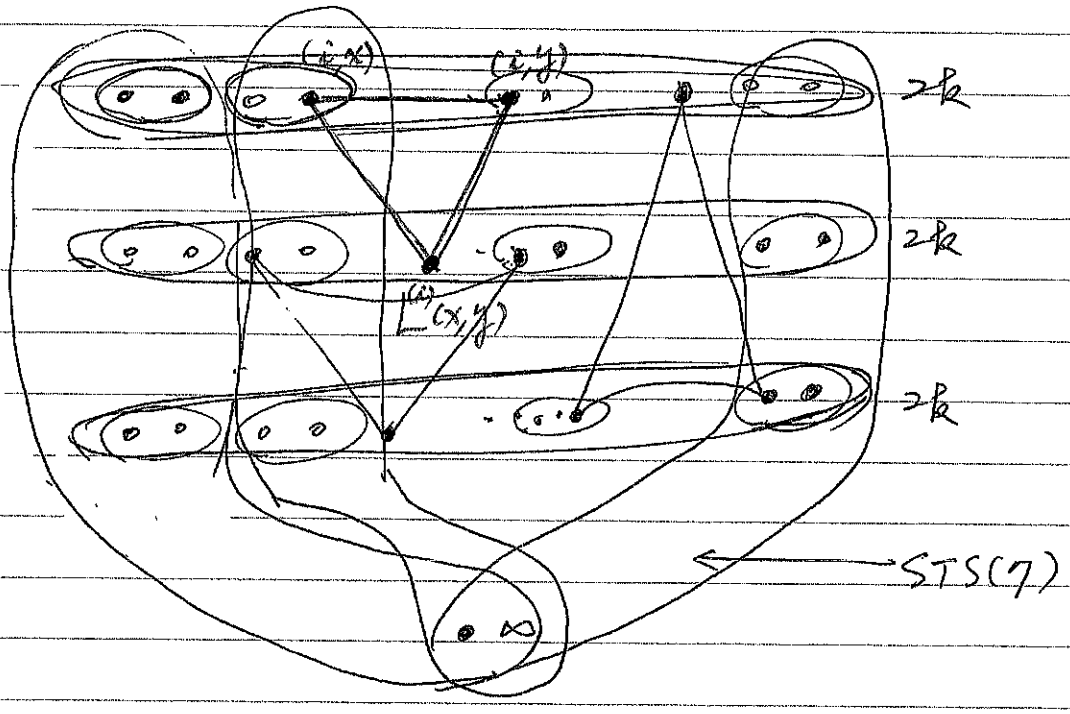
Case 1. $v \equiv 1 \pmod 6$, $v \geq 19$.

Let $v = 6k+1$, $k \geq 3$. Let $L^{(i)}$ be a commutative Latin square of order $2k$ defined on $\{(i,j) \mid i \in \mathbb{Z}_3 \text{ and } j \in \mathbb{Z}_{2k}\}$ with holes of size 2.

Now, let $(X, \mathcal{B})$ be a design with $X = \{\infty\} \cup (\mathbb{Z}_3 \times \mathbb{Z}_{2k})$, and $\mathcal{B}$ be defined as follows:

(a) $B \in \mathcal{B}$ if B is a block in an STS(7) defined on $\{\infty, (i,2h), (i,2h+1) \mid i \in \mathbb{Z}_3\}$ for each $0 \leq h \leq k-1$; and

(b) $\{(i,x),(i,y),(i+1, L^{(i)}(x,y))\} \in \mathcal{B}$ for all $i \in \mathbb{Z}_3$ and $x, y \in \mathbb{Z}_{2k}$ such that (i,x) and (i,y) are not in a 2×2 hole (The first component is taking modulo 3 and the 2nd component is taking modulo $2k$.)



It's left to check that $(X, \mathcal{B})$ is an $STS(v)$. First, we count $|\mathcal{B}|$. Since each entry outside the hole and in the upper part of $L^{(i)}$ gives a triple (block), we have $3 \cdot \frac{(2k)^2 - (2k)\cdot 2}{2} + 7\cdot k$ which is equal to $\frac{12k^2 - 12k + 14k}{2} = 6k^2 + k = \frac{1}{6}(6k+1)\cdot 6k = \frac{v(v-1)}{6}$. Hence, if each pair of two elements in X occurs, then the pair occurs at most once. So, we have to verify each pair of elements of X

does occur in a block of $\mathcal{B}$ defined above, (a) or (b). Clearly, if one of the elements is ∞, then $\{\infty, x\}$ occurs in the blocks defined in (a). On the other hand, consider (i_1, x) and (i_2, y) where $i_1, i_2 \in \mathbb{Z}_3$ and $x, y \in \mathbb{Z}_{2k}$. First, if they are in the holes of either $L^{(i_1)}$ or $L^{(i_2)}$ $(= L^{(i)})$, then they occur together in the block of (a). On the other hand, if they are not in the holes of $L^{(i)}$, then we have two cases to consider:

(i) $i_1 = i_2 = i$

Clearly, they occur together in $\{(i,x), (i,y), (i+1, L^{(i)}(x,y))\}$ in (b)

(ii) $i_1 \neq i_2$

Without loss of generality, let $i_2 \equiv i_1 + 1 \pmod 3$ and $i_1 = i$. Since there exists a $z \in \mathbb{Z}_{2k}$ such that $L^{(i)}(x,z) = y$, (i_1, x) and (i_2, y) will occur in $\{(i_1, x), (i_1, z), (i_2, y)\}$ of (b).

This concludes of proof. All STS(v)'s of order $v \equiv 1 \pmod 6$ have been constructed.

Next, let $v \equiv 3 \pmod 6$ and $X = \{\infty_1, \infty_2, \infty_3\} \cup (\mathbb{Z}_3 \times \mathbb{Z}_{2k})$.

The construction can be obtained similarly. The blocks in $\mathcal{B}$ can be defined as follows: (a) Use STS(9) instead of STS(7) when $\{\infty\}$ is replaced by $\{\infty_1, \infty_2, \infty_3\}$. Moreover, fix $\{\infty_1, \infty_2, \infty_3\}$ as a block for each STS(9). (b) Use the same construction.

Hence $|\mathcal{B}| = 1 + 11\cdot k + 3\cdot[(2k)^2 - 4k]/2 = (12k^2 + 10k + 2)/2 = 6k^2 + 5k + 1$

$= (2k+1)(3k+1) = \frac{(6k+3)(6k+2)}{6} = \frac{v(v-1)}{6}$. And the existence of every pair of distinct elements in X is similar. ■

(*) Exercise 3-1 (15 points)
Prove that an STS(v) exists for each $v \equiv 1$ or $3 \pmod 6$.
[Three different ways!!]

Note that the above construction was obtained not long time ago.

There are quite a few methods in constructing Steiner triple systems. One of the most "popular" one is called "cyclically constructing" method or in general, difference method. 5', 5''

Definition (Difference)

Let $X = \mathbb{Z}_n$. Then the difference of two distinct elements x and y in X is $\pm(x-y) \equiv_{def} \pm\|x-y\|$ such that $1 \le \|x-y\| \le \lfloor \frac{n}{2} \rfloor$. The differences obtained in a set S is the set of all difference of two distinct elements in S.

Example $S = \{0, 1, 3\} \subseteq \mathbb{Z}_7$

$\text{diff.}(S) = \{\pm 1, \pm 2, \pm 3\} \pmod 7$.

$= \{1, 2, 3, 4, 5, 6\}$.

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- If the difference of a and b is defined as $\min\{|a-b|, n-|a-b|\}$, then it is known as the circular difference of a and b or half-difference in short.
- $\{1,2,4\}$ in $\mathbb{Z}_7$ will provide three half-differences 1, 2, and 3. Clearly, in $\mathbb{Z}_n$, the set of half-differences will be $\{1,2,\cdots,\lfloor \frac{n}{2} \rfloor\}$.
- So, the set of half-differences in S is defined as $\underline{D_2(S)}$. Moreover, $|D_2(S)| \leq \binom{|S|}{2}$.
- Again, an equi-difference set S is the set whose $D_2(S)$ is of "smaller" cardinality. For example, in $\mathbb{Z}_8$, $D(\{1,2,3,4\}) = \{1,2,3\}$ and $D(\{0,2,4,6\}) = \{2,4\}$.

Definition (Difference set)

A set of k elements $D = \{a_1, a_2, \cdots, a_k\}$ in $\mathbb{Z}_v$ is called a (v,k,λ)-difference set if $\forall\, d \in \mathbb{Z}_v^*$ there are exactly λ ordered pairs (a_i, a_j), $a_i, a_j \in D$ such that $a_i - a_j \equiv d \pmod v$.

Difference Sets

Given a subset S of $\mathbb{Z}_n$. The set of differences in S, denoted by $D_2(S)$, is $\{a-b \pmod n \mid a,b \in S\}$. For example, if $n=7$ and $S=\{1,2,4\}$, then $D(S)=\{1,2,3,4,5,6\}$. Moreover, if $n=13$, $S=\{1,5,6,8,11\}$ then $D(S)=\{1,2,\cdots,12\}$.

- Observe that if $a,b \in S \subseteq \mathbb{Z}_n$, then $a-b \pmod n \in \mathbb{Z}_n^*$ provided $a \neq b$.
- If $|S|=s$, then $|D(S)| \leq 2\binom{s}{2}$ (provided $s \leq n$).
- A set S is called an equi-difference set if the elements of S form an arithmetic progression, i.e, $S=\{a, a+d, \cdots, a+(t-1)d\}$ where $a+(t-1)d \leq n$ and $d>0$.
- For example, $S=\{1,2,3,4\}$ is an equi-difference set in $\mathbb{Z}_8$. ($D(S)=\{1,2,3,5,6,7\}$)
- Note that an equi-difference set could create the minimum number of distinct differences among all other sets of the same cardinality.

Definition (Base blocks)

A collection $\mathcal{C}$, set of $= \mathbb{Z}_v$ of subsets of X is called a base blocks of a $2-(v,k,\lambda)$ design if the following conditions satisfied:

(1) Each set of $\mathcal{C}$ is of size k; and

(2) $\bigcup_{S\in\mathcal{C}} \text{diff}(S)$ contains each difference in $\pm\{1,2,\cdots,\lfloor\frac{n}{2}\rfloor\}$ exactly λ time

- Constructing design cyclically

Theorem 2 If $\mathcal{C}$ is a set of base blocks of a $2-(v,k,\lambda)$ design, then $(X,\mathcal{B}) = (\mathbb{Z}_v, \mathcal{B})$

$\mathcal{B} = \{ i+S \mid S\in\mathcal{C} \text{ and } i\in\mathbb{Z}_v\}$. (Note that if $S=\{x_1,x_2,\cdots,x_k\}$, then $i+S = \{x_1+i, x_2+i, \cdots, x_k+i\} \pmod v$.)

Example $X=\mathbb{Z}_7$, $\mathcal{C}=\{\{0,1,3\}\}$ is a set of base block of an STS(7).

Example $X=\mathbb{Z}_{15}$, $\mathcal{C}=\{\{0,3,4\},\{0,2,8\},\{0,5,10\}\}$ is a set of base blocks of an STS(15). Note that $\{0,3,4\}$ and $\{0,2,8\}$ generate 15 blocks resp. and $\{0,5,10\}$ generates 5 blocks.

Definition (Full orbit and short orbit)

A base block is of full orbit (short orbit) if the block generates v blocks in $(\mathbb{Z}_v, \mathcal{B})$ resp.. (less than v blocks)

Definition (Cyclic design)

A design is called a cyclic design if the design can be obtain by using a set of base blocks.

(*) The above STS(7) and STS(15) are cyclic Steiner Triple systems.

(**) No cyclic STS(9) exists!

Theorem 3 : A cyclic Steiner triple system of order $v \neq 9$ exists.

In order to prove the above theorem, we need to find a set of base blocks for each order $v \equiv 1$ or $3 \pmod 6$. Therefore, a systematic construction should be obtained.

Definition (Skolem sequences)

A Skolem sequence of order n is a sequence of length $2n$ $(a_1, a_2, \cdots, a_{2n})$ such that each of the elements in $\{1, 2, \cdots, n\}$ occurs exactly twice, moreover, the indexes of "i" occurred exactly "i" apart, i.e., $\forall i \in \{1, 2, \cdots, n\}$, if $a_t = i$, then either $a_{t+i} = i$ or $a_{t-i} = i$ (not both).

Example. $n = 4$, $\langle 1, 1, 3, 4, 2, 3, 2, 4 \rangle$.
$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8$

$n=5$, $\langle 1,1,3,4,5,3,2,4,2,5\rangle$.

Theorem 4. A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$. (Type A)

Exercise 3-2. (5 points) Find a proof of Theorem 4 and 5, resp.

For $n \equiv 2$ or $3 \pmod 4$, we can obtain an extended Skolem seq by using one extra bit. For example, $n=2$, $\langle 1,1,2,-,2\rangle$, $n=3$, $\langle 1,1,2,3,2,-,3\rangle$. (Use $\langle a_1, a_2, \cdots, a_{2n-1}, \overline{a_{2n}}, a_{2n+1}\rangle$.)

Theorem 5 An extended Skolem sequence of order n exists if and only if $n \equiv 2$ or $3 \pmod 4$. (Type B)

For convenience, we can also use their indices of a Skolem sequence or an extended Skolem sequence to represent the sequence. For example, $\langle 1,1,3,4,5,3,2,4,2,5\rangle =_{def} \{\{1,2\},\{7,9\},\{3,6\},\{4,8\},\{5,10\}\}$, and $\langle 1,1,2,3,2,-,3\rangle =_{def} \{\{1,2\},\{3,5\},\{4,7\}\}$. Therefore, they are partitions of $[1,10]$ and $[1,7]\setminus\{6\}$ into five 2-subsets and three 2-subsets respectively.

For convenience, we shall use set-notation for Skolemn sequence

So, we have a revised definition for Skolemn seq.

Definition

A Skolemn sequence of order n is a partition of $[1,2n]$ into 2-subsets $\{\{a_i,b_i\} \mid i=1,2,\cdots,n\}$ such that $|a_i-b_i|=i$, $1\le i\le$

An extended Skolemn sequence of order n is a partition of $[1,2n+1]\setminus\{2n\}$ into 2-subsets $\{\{a_i,b_i\} \mid i=1,2,\cdots,n\}$ such that $|a_i-b_i|=i$ for $i=1,2,\cdots,n$.

Fact (9/) A Skolemn sequence of order n exists if $n\equiv 0$ or $1 \pmod 4$

An extended Skolemn sequence exists if $n\equiv 2$ or $3 \pmod 4$.

Fact For each $d\in\mathbb{N}$, $[1+d, 2n+d]$, a Skolemn sequence of order n exists if $n\equiv 0$ or $1 \pmod 4$. This is also true for extended Skolemn sequence on $[1+d, 2n+d+1]\setminus\{2n+d\}$.

In fact, there are quite a few modified sequences by using the above two sequences.

Skolem Sequences and Extended Skolem Sequences

(Constructions)

$n \equiv 0 \pmod 4$ $\quad n \geq 6$

Let $n = 4k$.

	Pairs ⇓	Differences ⇓
$r = 1, 2, \cdots, 2k$	$\{4k+r-1, 8k-r+1\}$	$\{4k-2r+2\}^{2k}_{1}$ $\{2, 4, \cdots, 4k\}$
$r = 1, 2, \cdots, k-2$	$\{r, 4k-r-1\}$	$\{4k-2r-1\}$ $\{2k+3, \cdots, 4k-3\}$
$r = 1, 2, \cdots, k-2$	$\{k+r+1, 3k-r\}$	$\{2k-2r-1\}$ $\{3, \cdots, 2k-3\}$
	$\{k-1, 3k\}, \{k, k+1\}, \{2k, 4k-1\}, \{2k+1, 6k\}$	$\{2k+1, 1, 2k-1, 4k-1\}$

$n \equiv 1 \pmod 4$

Let $n = 4k+1$

	Pairs	Differences
$r = 1, 2, \cdots, 2k$	$\{4k+r-1, 8k-r+3\}$	Check yourself!
$r = 1, 2, \cdots, k$	$\{r, 4k-r+1\}$	
$r = 1, 2, \cdots, k-2$	$\{k+r+2, 3k-r+1\}$	
	$\{k+1, k+2\}, \{2k+1, 6k+2\}$	
	$\{2k+2, 4k+1\}$	

$n \equiv 2 \pmod 4$ $\quad (n \geq 6)$

Let $n = 4k+2$.

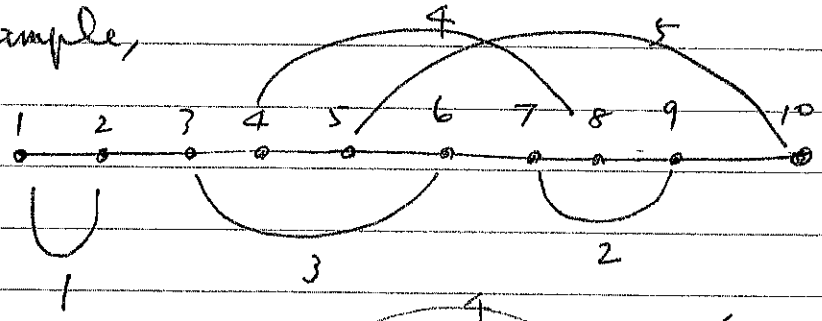
	Pairs	Differences
$r = 1, 2, \cdots, 2k$	$\{r, 4k-r+2\}$	Check!
$r = 1, 2, \cdots, k-1$	$\{4k+r+3, 8k-r+4\}$	
$r = 1, 2, \cdots, k-1$	$\{5k+r+2, 7k-r+3\}$	
	$\{2k+1, 6k+2\}, \{4k+2, 6k+3\}$	
	$\{4k+3, 8k+5\}, \{7k+3, 7k+4\}$	

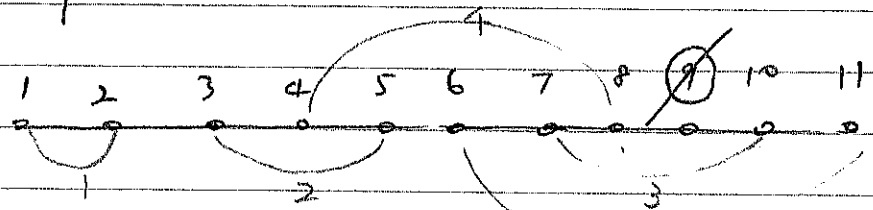
$(n \geq 6)$

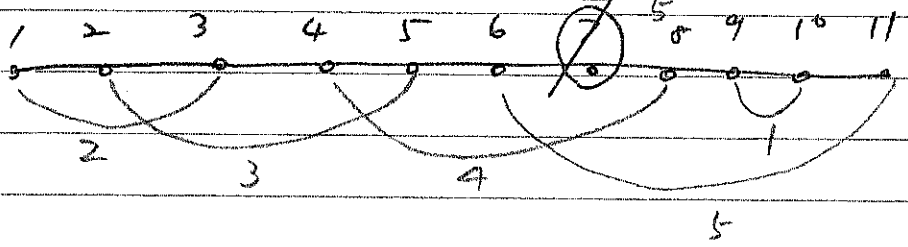
$n \equiv 3 \pmod 4$

	Pairs	Differences (Check!)
$n = 4k-1$	$\{4k+r, 8k-r-2\}$, $r = 1, 2, \cdots, 2k-2$,	
	$\{r, 4k-r-1\}$, $r = 1, 2, \cdots, k-2$,	
	$\{k+r+1, 3k-r\}$, $r = 1, 2, \cdots, k-2$,	
	$\{k-1, 3k\}, \{k, k+1\}, \{2k, 4k-1\}$	
	$\{2k+1, 6k-1\}, \{4k, 8k-1\}$	

For example,



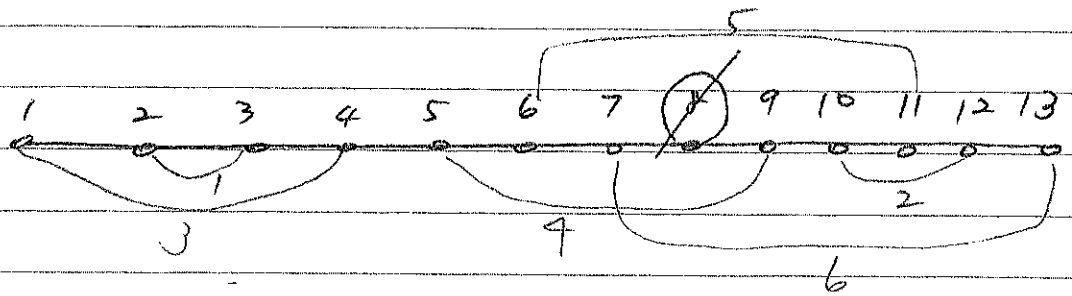




(*) Fact $[1, 2m+1] \setminus \{k\}$ can be partitioned into 2-subsets $\{\{a_i, b_i\} \mid i = 1, 2, \cdots, n\}$ such that $|a_i - b_i| = i$ for each $1 \le i \le n$ provided,

(a) $n \equiv 0$ or $1 \pmod 4$ and $k \equiv 1 \pmod 2$; and

(b) $n \equiv 2$ or $3 \pmod 4$ and $k \equiv 0 \pmod 2$.



Bonus (5 points) Prove the above fact.

NO. 11
DATE

Theorem 6 A cyclic STS(v) exists if and only if $v \neq 9$ and $v \equiv 1$ or $3 \pmod 6$.

Proof:

Case 1 $v \equiv 1 \pmod 6$

For convenience, let $v = 6k+1$ and we consider the "half" difference. Then, the set of differences is $\{1, 2, \cdots, 3k\}$. Note that the difference $3k$ is the same as the difference $3k+1$. Hence, by using Skolemn sequences of type A or type B, $\{k+1, k+2, \cdots, 3k\}$ or $\{k+1, k+2, \cdots, 3k-1, 3k+1\}$ can be partitioned into 2-subsets $\{\{a_i, b_i\}$ (respectively) $i = 1, 2, \cdots, k\}$, such that $|a_i - b_i| = i$, $i = 1, 2, \cdots, k$. Now, we have k difference-triples $\{i, a_i, b_i\}$, $i = 1, 2, \cdots, k$. This implies that we can find k base blocks $\{0, i, i+a_i\}$ ($a_i < b_i$) for $i = 1, 2, \cdots, k$ ($i + a_i = b_i$). And, thus we have a cyclic STS(v).

For your ref.

Example, $v = 19$, differences are $1, 2, 3, \cdots, 9$.

$4, 5, 6, 7, 8, 10 \Rightarrow$ Difference triples are $\{1, 4, 5\}, \{2, 6, 8\}, \{3, 7, 10\}$,

Base blocks: $\{0, 1, 5\}, \{0, 2, 8\}, \{0, 3, 10\}$.

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Case 2 $v \equiv 3 \pmod 6$

Let $v = 6k+3$ and the set of half differences is $\{1, 2, \cdots, 3k+$

First, we delet $2k+1$ from the above set. Hence, the set of differences is $\{1, 2, \cdots, k, k+1, k+2, \cdots, 2k, 2k+2, \cdots, 3k+1\}$. It remains to show that $\{k+1, k+2, \cdots, 2k, 2k+2, \cdots, 3k+1\}$ (or $3k+2$) can be partitioned into 2-subsets $\{a_i, b_i\}$ such that $|a_i - b_i| = b_i - a_i = i$ $(b_i > a_i)$ for $i = 1, 2, \cdots, k$. Again, it can be done by using either $3k+1$ or $3k+2$ in respective cases. In fact, it is up to the relationship beteen k and v. If $k \equiv 1$ or $2 \pmod 4$, then we partition $\{k+1, k+2, \cdots, 3k, 3k+1\} \setminus \{2k+1\}$. On the other hand, if $k \equiv 3$ or $4 \pmod 4$ then we partition $\{k+1, k+2, \cdots, 3k+1\} \setminus \{2k+1\}$ into 2-subsets satisfying $|a_i - b_i| = b_i - a_i = i$ $(b_i > a_i)$. ∎

(into suitable 2-sets provided $k \equiv 0$ or $1 \pmod 4$)

For your ref.

Example, $v = 15$, $\{k+1, \cdots, 3k+1\} \setminus \{2k+1\} = \{3, 4, 6, 7\}$ (or 8)

Difference triples are $\{1, 3, 4\}, \{2, 6, 8\}, \{5\}$.

Base blocks are: $\{0, 5, 10\}, \{0, 1, 4\}, \{0, 2, 8\}$.

NO. 13

DATE

$v = 21$, $D = \{4, 5, 6, 8, 9, 10\}$

Triples are $\{1, 4, 5\}, \{2, 8, 10\}, \{3, 6, 9\}, \{7\}$.

Base blocks are $\{0, 7, 14\}, \{0, 1, 5\}, \{0, 2, 10\}, \{0, 3, 9\}$.

$v = 27$, $6k+3$, $k = 4$

1, 2, 3, 4, 5 || 6, 7, 8, ~~9~~, 10, 11, 12, 13

$v = 33$, $6k+3$, $k = 5$

1, 2, 3, 4, 5 || 6, 7, 8, 9, 10, ~~11~~, 12, 13, 14, 15, ~~16~~, 17

$v = 39$, $6k+3$, $k = 6$

1, 2, 3, 4, 5, 6 || 7, 8, 9, 10, 11, 12, ~~13~~, 14, 15, 16, 17, 18, ~~19~~, 20

$v = 45$, $6k+3$, $k = 7$

1, 2, 3, 4, 5, 6, 7 || 8, 9, 10, 11, 12, 13, 14, ~~15~~, 16, 17, 18, 19, 20, 21, 22

$v = 51$, $6k+3$, $k = 8$

1, 2, 3, 4, 5, 6, 7, ~~8~~ || 9, 10, 11, 12, 13, 14, 15, 16, ~~17~~, 18, 19, 20, 21, 22, 23, 24, 25