Problems and Conjectures in Graph Theory

作者：傅恆霖 教授
編排者：研究生 陳冠帆

Problems and Conjectures in Graph Theory

Theorem 1.1 : Let D ba a digraph with adjacency matrix A. Then D has a directed even cycle if and only if $per(I + A) \neq det(I + A)$, where "per" denotes the permanent and "det" the determinant.

Problem 1 : Is there a polynomial-time algorithm for finding a directed even cycle in a digraph? (D. H. Younger, 1981)

Problem 1′ : Can we prove that finding a directed even cycle in a digraph is NP-complete?

Best result: Thomassen(1993), algorithm (p) for planar digraphs.

Theorem 2.1 : Let G be a k-connected graph, where $k \geq 2$, and let S be a set of k vertices in G. Then G contains a cycle C that includes every vertex of S. (Dirac, 1960)

Theorem 2.2 : Let G be a simple k-connected graph of minimum degree δ, where $k \geq 2$, and let S be a set of k vertices of G. Then G contains either a cycle of length at least 2δ that includes every vertex of S or a hamiltonian cycle. (Egawa et al, 1991).

Conjecture 2 : In a k-connected graph, where $k \geq 2$, any two longest cycles have at least k vertices in common.

Conjecture 2′ : Let G be a k-connected graph, where $k \geq 2$, and let S be a set of k edges in G that induce a linear forest. Then there is a cycle C of G that includes every edge of S, unless k is odd and S is an edge cut of G.

Theorem 2.3 : Let G be a k-connected graph, where $k \geq 2$, and let S be a set of "$k - 1$" edges in G that induces a linear forest. Then G has a cycle that includes every edge of S. (Woodall, 1977)

Definition : (cyclically k-edge-connected) A connected graph that contains two disjoint cycles is cyclically k**-edge connected** if it has no edge cut S of fewer than k edges such that both components of $G - S$ contain cycles.

Conjecture 3 : Let G be a 3-regular cyclically k-edge-connected graph and let S be a set of $k-1$ edges that induces a linear forest. Then there is a cycle of G that includes every edge of S.

Theorem 4 : (Chv*á*tal-Erd*ö*s) Let G be a graph on at least these vertices with independence (stability) number α and connectivity κ, where $\alpha \leq \kappa$. Then G is hamitlonian.

Definition : (Pancyclic) A graph is pancyclic if the graph contains a cycle of length k for each $3 \leq k \leq |V(G)|$.

Conjecture 4 : If $\alpha \leq k-1$, then G is pancyclic. (Jackson and Ordaz, 1990)

Definition : (σ_k) $\sigma_k = min\{\sum_{x \in S} deg(x) |$ S is an independent set of G with k vertices $\}$.

Theorem 5.1 : Let G be a simple k-connected graph of order n such that $\sigma_{k+1} \geq n + k(k-1)$, and let C be a longest cycle in G. Then $G - C$ contains no complete subgraph of order k. (Bondy, 1981)

Theorem 5.2 : Let G be a simple graph of connectivity κ on n vertices such that $\sigma_3 \geq n + k$, where $\kappa \geq 2$. Then G is hamiltonian. (Bauer et al. 1989)

Conjecture 5 : Let G be a simple k-connected graph of order n such that $\sigma_{k-1} \geq n + k(k+1)$, and let C be a longest cycle in G. Then $G - C$ contains no path of length $k-1$.

Definition : (Toughness) Let t ba a positive real number. A graph G is t-tough if $c(G - S) \leq |S|/t$ for every vertex cut set S of G. The toughness of a graph G is the "largest value" of t for which G is t-tough.

Best result: We may construct a t-tough graph which is not hamiltonian for $t < 2$. (Enomoto et al. 1985)

Conjecture 6.1 : Every 2-tough graph is hamiltonian. (Chv*á*tal, 1973)

Definition : (k-tree) A k-tree of a graph G is a spanning tree of G in which each vertex is of degree at most k.

Theorem 6.2 : Let k be a positive integer. Then every $\frac{1}{k}$-tough graph has a $(k+2)$-tree.

Definition : (k-walk) A k-walk is a spanning walk that passes through each vertex no more that k times.

Conjecture 6.3 : Let k be a positive integer. Then every $\frac{1}{k}$-tough graph has a $(k+1)$-walk.

Known result : If $\sigma_{k+1} \geq n$, then G has a k-walk. (Jackson and Wormald, 1990)

Theorem 7.1 : (Tutte) Every 4-connected planar graph is hamiltonian.

Theorem 7.2 : (Gao and Richter, 1994) Every 3-connected planar graph has a 2-walk.

Theorem 7.3 : (Robinson and Wormald, 1991) For every integer $d \geq 3$, almost all d-regular graphs are hamiltonian.

Theorem 7.4 : (Richmond et al., 1985) Almost all 3-connected 3-regular planar graphs are not hamiltonian.

Theorem 7.5 : (Jackson) Let G be a 2-connected d-regular on n vertices, where $d \geq \frac{n}{3}$. Then G is hamiltonian.

Conjecture 7.1 : Let G be a 2-connected d-regular graph on n vertices, where $d \geq \frac{n}{r}$, $r \geq 3$, and n is sufficiently large. Then G contains a cycle of length at least $2n/(r-1)$. (Bondy, 1978)

Conjecture 7.2 : Let G be a 3-connected d-regular graph of order n such that $d \geq \frac{n}{4}$. Then G is hamiltonian unless G is either the Petersen graph or the graph obtained from it by expanding one vertex to a triangle. (B. Jackson, 1989)

Best result : (Fan, 1985; Jung, 1984) Let G be a 3-connected d-regular graph. Then G contains either a cycle of length at least $3d$ or a hamiltonian cycle.

Conjecture 8 : Let G be a 3-regular cyclically 4-edge-connected graph of order n. Then G contains a cycle of length at least cn where c is a positive number. (Thomassen et al., 1986, et al.)

Theorem 9 : Let G be a 3-connected 3-regular graph , and let S be a set of nine vertices of G. Then G has a cycle which includes every vertex of S. (Aolton et al., 1982; Kelmans and Lomonosov, 1982)

Problem 9 : Dose there exit a constant $c > 1$ such that , for every d-regular d-connected graph G and for every set S of $\lfloor cd \rfloor$ vertices of G, where $d \geq 2$, there is a cycle of G that includes every vertex of S.

Problem 10 : Dose every connected vertex-transitive have a hamiltonian cycle?

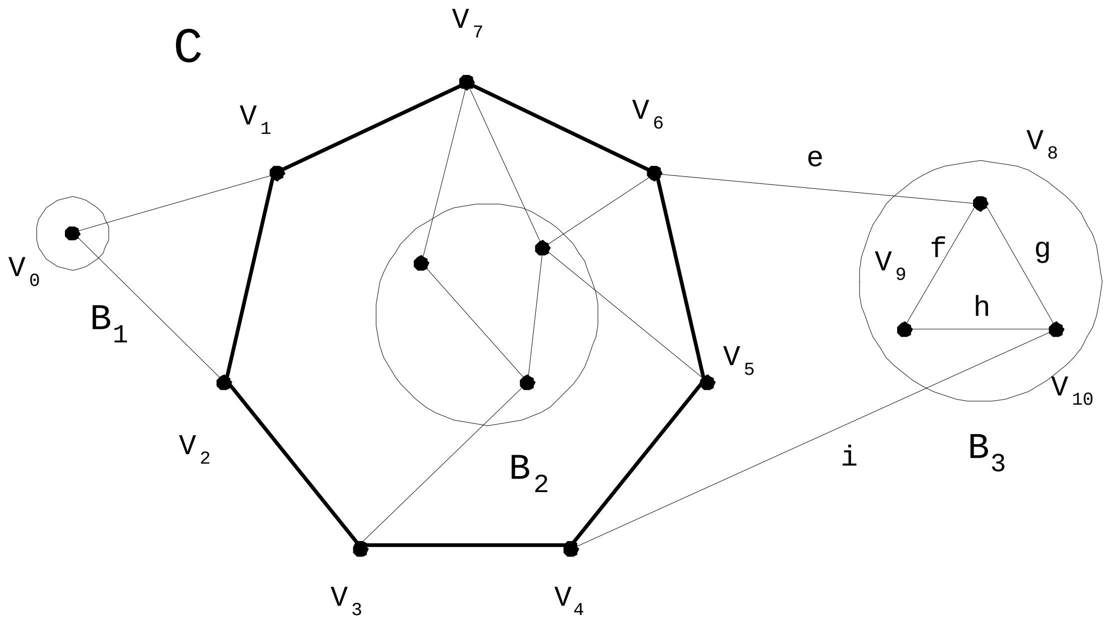
Definition : (Cayley graph) Let Γ be a group and X be a set of elements of Γ. The digraph $D(\Gamma, X)$ with vertex set Γ and arc set $\{(g, gx)|g \in \Gamma, x \in X\}$ is the Cayley digraph of Γ with respect to X. If for each $x \in X$, $x^{-1} = x$, then $D(\Gamma, X)$ is the associated digraph of a simple graph $D(\Gamma, X)$, called the Cayley graph of Γ with respect to X.

Conjecture 11 : Every Cayley graph is hamiltonian. (T. D. Parsons)

Conjecture 12 : Every 4-connected line graph is hamiltonian. (Thomassen, 1986)

Conjecture 13 : Every 4-connected claw-free graph is hamiltonian. (A graph is claw-free if the graph contains no $K_{1,3}$ as an induced subgraph.)

Definition : (Bridges of subgraphs) Let G be a graph and H a subgraph of G. Consider the equivalence relation $\sim$ on $E(G) \setminus E(H)$ defined by $e \sim f$ if and only if there is a walk internally-disjoint vertex from H whose first edge is e and whose last edge is f. The subgraphs of G induced by the equivalent classes under this relation are the "bridges" of H in G; isolated vertices of G in $G - H$ are also regarded as bridges of H. Abridge having at most one edge is "degenerate"; all other bridges are "proper". A chord of H is a degenerate bridge linking two distinct vertices of H. If B is a bridge of H in G, the elements of $V(B) \setminus V(H)$ are the internal vertices of B and the elements of $V(B) \cap V(H)$ the "vertices of attachment" of B to H. The edges of B incident to its vertices of attachment are its "edges of attachment" to H.



Bridges of C : $V(B_3) = \{v_4, v_6, v_8, v_9, v_{10}\}$, $E(B_3) = \{e, f, g, h, i\}$.

Definition : (Tutte cycle) A Tutte cycle is a cycle of length at least 4 each of whose bridges has at most three vertices of attachment.

Conjecture 14 : Every 2-connected claw-free graph on at least four vertices contains a Tutte cycle. (Jackson 1992)

Conjecture 14′ : Every simple 2-edge-connected graph on at least four edges contains a connected even subgraph on at least four edges each of whose bridges has at most three edges of attachment.

Definition : (Oriented graph and tournament) The underlying graph $G(D)$ of a digraph D is the graph obtained from D by ignoring the orientation of its edges. An "oriented" graph is a digraph whose underlying graph is simple. A "tournament" is a digraph whose underlying graph is simple and complete. A strong digraph is also known as a strongly connected graph in which any tow vertices can be connected b a directed path.

Conjecture 15 : Let D be a strong oriented graph with minimum indegree and outdegree at least k. Then D has a directed cycle of length at least $2k+1$. (Jackson, 1980)

Conjecture 16 : Let D be an oriented graph of minimum degree δ^+ on n vertices, where $\delta^+ \geq (3n-2)/8$. Then D contains a directed hamiltonian cycle. (H*ä*ggkvist, 1993)

Conjecture 17 : Every k-diregular oriented graph on at most $4k+1$ vertices, where $k \neq 2$, contains a directed hamiltonian cycle. (Jackson, 1981)

Problem 18 : Is there a nonhamiltonian oriented graph in which the reversal of any edge results in a hamiltonian digraph? (Thomassen, 1987)

Conjecture 19 : For each fixed k, there exists a polynomial-time algorithm for deciding if there ia a directed hamiltonian cycle through k prescribed edges in a tournament. (Bang-Jensen and Thomasen, 1992)

Definition : (Path partition) A partition of a graph G is a family of disjoint subgraphs whose vertices partition $V(G)$. A "path partition" is a partition into paths, and the minimum number of paths in a path partition of a graph G, path partition number, is denoted by $\pi(G)$.

Theorem 20 : (The Gallai-Milgram Theorem)

Let D be a digraph with independence number α. Then D admits a partition into at most α directed paths. That is, $\pi(D) \leq \alpha(D)$.

Conjecture 20 : Let D be a strong digraph with independence number α. Then D admits a vertex covering by at most α directed cycles. (Gallai, 1964)

(Note: A vertex covering of a graph or digraph is a family of subgraphs whose union spans the graph or digraph, i.e., the union of the vertex-sets of the subgraphs is the vertex-set of the graph or digraph.)

Conjecture 21 : Let D be a strong digraph with independence number α, where $\alpha \geq 2$. Then D admits a partition into at most $\alpha - 1$ directed paths.

Best results :

Theorem 21 : (Chen and Manalastas, 1983 ; Bondy, 1995)

Let D be a strong digraph with $\alpha(G) = 2$. Then D admits a vertex covering by two directed cycles $\{C_1, C_2\}$, where $C_1 \cap C_2$ is either empty or a directed path of D.

Corollary 21 : Let D be a strong digraph with $\alpha(D) = 2$. Then D contains a directed hamiltonian path.

Definition : (Longest directed path) The number of vertices in a longest directed path of a digraph D is denoted by $\lambda(D)$.

Theorem 22 : (Gallai-Roy Theorem)

Let D be a digraph with chromatic number x. Then D contains a directed path of length at least $x - 1$. That is , $\lambda(D) \geq x(D)$.

Conjecture 22 : Let D be a nontrivial digraph and $\lambda(D) = \lambda_1 + \lambda_2$, where λ_1 and λ_2 are positive integers. Then there is a partition (V_1, V_2) of $V(D)$ such that $\lambda(D[V_i]) = \lambda_i$, $i = 1, 2$. (Laborde et al., 1983)

Conjecture 23 : For any positive integer k, there exists a digraph D with independence number k such that $\alpha(D \setminus \bigcup_{i=1}^{k-1} P_i) = \alpha(D)$ for any family $\{P_1, P_2, \ ... \ , P_{k-1}\}$ of $k - 1$ disjoint directed paths of D.

Problem 24 : Is there a maximally nonhamiltonian graph of girth g for all $g \geq 3$. (Horák and Širáň, 1986)

Best result : There exist infinite families of maximally nonhamiltonian graphs of girths five and seven.

Conjecture 25 : Let D be a d-diregular digraph on n vertices without loops and multi-arcs. Then D has girth at most $\lceil n/a \rceil$.

(Note : A digraph D is strict if D contains no loops and multi-arcs.)

Conjecture 26 : Let D be a strict digraph of minimum outdegree δ^+ on n vertices. Then D has girth at most $\lceil n/\delta^+ \rceil$.

Best result :

1. $\delta^+ \leq 5$, the conjecture holds. (Hoáng and Reed, 1987)

2. Near-optimal upper bound $\frac{n}{\delta^+} + 2500$. (Chvátal and Szemerédi, 1983)

Conjecture 26′ : Let $\delta^+ \geq \frac{n}{3}$. Then D has a directed triangle.

Conjecture 27 : Let C be a longest cycle in a 3-connected graph. Then G has at least one chord. (Thomassen, 1976; Alspach and Godsil, 1985; Fleischner and Jackson 1989.)

Conjecture 28 : Let G be a 3-connected cyclically 4-edge-connected graph. Then G has a dominating cycle. (As above.)

Best result : The conjecture holds for planar graphs.

(Hint : C is a dominating cycle of a graph G if $V(G)\backslash V(C)$ is an independent set.)

Theorem 29 : Every simple graph of minimum degree at least 3 contains a cycle of length $3t$. (Chen and Saito, 1994)

Conjecture 29 : Every simple graph of minimum degree at least k, where $k \geq 3$, contains a cycle of length kt. (N. Dean, 1993)

Best result : The conjecture holds for $k = 4$. (N. Dean, 1993)

Conjecture 30 : Every simple 2-connected nonbipartite graph of minimum degree at least $k + 2$ contains cycles of all lengths $l \pmod k$.

Problem 31 : For each $l \geq 2$ and for infinitely many values of n, construct a simple graph on n vertices and $cn^{1+\frac{1}{e}}$ edges, where c is a constant depending only on l, containing no $2l$-cycle.

Best results :

1. $l = 2$. (Brown, 1966; Erdös et al. 1966; Füredi, 1983)

2. $l = 3, 5$. (Benson, 1966) Both use finite geometries.

3. The cases in digraphs are settled by Haggkvist and Thomassen, 1976.

4. Reciman (1985) use $\sum_{v \in V} \binom{d(v)}{2} > \binom{n}{2}$ to show that $ex(n; C_4) \leq n^{\frac{2}{3}}/2 + n/4$.

Conjecture 32 : Let G be a simple hamiltonian bipartite graph on $2n$ vertices and at least $\frac{n^2}{4} + n + 1$ edges. Then G is bipancyclic, i.e., G contains cycles of every length $2l$, $2 \leq l \leq n$. (Mitchen and Schmeichel, 1985)

Best result : It is correct if we have at least $\frac{n^2}{2}$ edges.

Conjecture 33 : Let G be a simple hamiltonian bipartite graph of minimum degree δ on $2n$ vertices, where $\delta^2 - \delta \geq n$. Then G is bipancyclic. (Mitchen and Schmeichel, 1985)

Conjecture 34 : The number of cycles in a 3-connected 3-regular graph on n vertices is superpolynomial in n, i.e., the number of cycles is always moderately large.

Conjecture 35 : Let D be a strict digraph with at least one directed cycle. Then D has an edge whose reversal reduces the total number of directed cycles.

Best result : If D has multi-arcs, then it is possible. (Grinberg, 1987; Thomassen, 1987).

Problem 36 : Is there a diregular tournament on n vertices with at least $(n-1)!/2^n$ directed hamiltonian cycles?

Problem 37 : Is there a uniquely hamiltonian simple graph of minimum degree 4. (Entringer and Swart, 1980) (Note : "min. degree 3" is possible.)

Conjecture 37′ : Every 4-regular hamiltonian simple graph has at least two distinct hamiltonian cycles. (Sheehan, 1975)

Theorem 38 : (Corrádi and Hajnal, 1963)

Let k be a positive integer and G a simple graph of minimum degree δ on n vertices, where $\delta \geq 2k$ and $n \geq 3k$. Then G has k disjoint cycles.

Conjecture 38 : Let k be a positive integer and D a strict digraph of minimum outdegree $\delta^+ \geq 2k-1$. Then D has k disjoint directed cycles. (Bermond and Thomassen, 1981)

Theorem 39 : (Lucchesi and Younger, 1978)

Let D be a planar digraph. Then the maximum number of edge-disjoint directed cycles in D is equal to the minimum number of arcs of D whose deletion destroys all directed cycles.

Conjecture 39 : Let D be a planar digraph. Then the minimum number of edges in a directed cycle of D (the girth of D) is equal to the maximum number of disjoint sets of edges of D the deletion of each of which destroys all directed cycles. (Woodall, 1978)

Theorem 40.1 : In any digraph, there is a directed path or directed cycle whose arcs meet every maximal arc cut.

Theorem 40.2 : Let D be a digraph. Then either D has two arc-disjoint cycles or there is a set of at most three arcs in D meeting all directed cycles.

Theorem 40.3 : (Jackson, 1979)

Every simple d-regular graph on n vertices, where $d \geq (n-1)/2$ and $n \geq 14$, contains at least $\lfloor (3d - n + 1)/6 \rfloor$ edge-disjoint hamiltonian cycles.

Conjecture 40 : Let G be a simple graph and k is a positive integer. Then either G has $k+1$ edge-disjoint triangles or there is a set of at most $2k$ edges meeting all triangles. (Tuza, 1984)

Conjecture 41 : Let G be a simple 2-connected graph on n vertices. Then $E(G)$ can be covered by at most $(2n-1)/3$ cycles. (Pyber, 1985)

Definition : Let C be a cycle covering of G. Then the length of $\boldsymbol{C}$, $||\boldsymbol{C}||$ is defined to be $\sum_{C \in \boldsymbol{C}} e(\boldsymbol{C})$ and $cc(G) = min\{||\boldsymbol{C}|| \; |\boldsymbol{C}$ is a cycle covering of $G\}$.

Conjecture 42 : The problem of determing $cc(G) \leq k$ is NP-complete. (Thomassen)

Conjecture 43 : Every 2-edge-connected graph on e edges admits a cycle covering $\boldsymbol{C}$ such that $||\boldsymbol{C}|| \leq 7e/5$.

(Note : For Petersen graph G, $cc(G) = 7e/5$.)

Best result : (Bermond et al., 1983) $||\boldsymbol{C}|| \leq 5e/3$.

Conjecture 44 : Let G be a simple 2-connected 3-regular graph on n vertices. Then G admits a decomposition into at most $n/10$ paths. (Reed, 1989)

Best result : $n/9$ paths.

Conjecture 45 : Let n be a positive integer, $n \geq 3$, and let $n_1 + n_2 + \ ... \ + n_k$, where $n_i \geq 3$, be a partition of n into k parts, s of which are odd. If G is a simple graph of minimum degree δ on n vertices, where $\delta \geq (n+s)/2$, then G admits a decomposition into k cycles, of length $n_1, n_2, \ ... \ , n_k$ respectively. (El-Zahar, 1984)

Best result : $k = 2$ was verified by El-Zahar.

Conjecture 46 : Let G be a simple graph of minimum degree δ on n vertices, where $\delta \geq 2n/3$. Then G contains the square of a hamiltonian cycle. (L. P*ó*sa, 1964)

Best result :

1. If $n(\varepsilon)$ is sufficiently large and $\delta \geq (2+\varepsilon)n/3$, then G contains the square of a hamiltonian cycle.
2. Let G be a simple graph of order n, where $\delta(G) \geq (2n-1)/3$. Then G contains the square of a hamiltonia path. (Fan and Kierstead, 1995)

Theorem 47.1 : (Toida, 1973)
An even graph admits an odd number of cycle decompositions, i.e., the graph can be decomposed into an odd number of cycles.
(Note : The theorem can be proved by using Thomassen's Lemma.)

Thomassen's Lemma : Let G be a connected graph of order $n \geq 2$, and let $x \in V(G)$. Then the number of longest x-path of G that terminate in a vertex of even degree is even. (An x-path is path with tail x.)

Theorem 47.2 : Let G be a 2-connected planar even graph on an even number of edges. Then G can be decomposed into even cycles. (Seymour, 1981)

Conjecture 47.1 : Every simple even graph on n vertices admits a decomposition into at most $(n-1)/2$ cycles. (Haj*ó*s)

Conjecture 47.2 : Every simple connected graph on n vertices admits a decomposition into at most $(n+1)/2$ paths. (T. Gallai)

Best results :

1. Tao (1984) verified Conjecture 47.1 for planar graph.
2. Lor*á*sz proved that every simple connected graph on n vertices admits a decomposition into at most $\frac{n}{2}$ paths and cycles. So, if the graph is odd, Conjecture 47.2 is verified.

Conjecture 48 : Every simple graph on n vertices admits a decomposition into at most cn cycles and edges where c is an absolute constant. (Erdös and Gallai, 1966)

Conjecture 49 : Let D ba an eulerian oriented graph on n vertices. Then D admits a decomposition into at most $2n/3$ directed cycles. (Dean, 1986)

Definition : (Excess) $ex(D) := \sum_{v \in V} max\{d^+(v) - d^-(v), o\}$.

Conjecture 50 : Let D be a regular oriented graph of odd degree. Then D admits a decomposition into $ex(D)$ directed paths.

Conjecture 51 : Let G be a simple even graph of minimum degree δ on n vertices and $m = 3t$ edges, where $\delta \geq \frac{3}{4}n$. Then G admits a decomposition into triangles provided n is sufficiently large. (Nash-Williams, 1970)

> **Best result :** (Gustavsson, 1991) The conjecture holds for $\delta \geq (1-10^{-24})n$.

Theorem 52 : (Hilton, 1984) Let s and n be positive integers $s \leq 2n$, let G and H be complete graphs of order s and $2n+1$, respectively, and let $\{E_1, E_2, \ldots, E_n\}$ be an n-edge coloring of G. Then, there is a hamiltonian decomposition $\{H_1, H_2, \ldots, H_n\}$ of H such that $E_i \leq H_i$, $1 \leq i \leq n$ if and only if the spanning subgraph of G with edge set E_i is a linear forest with at most $2n + r - 1$ components, $1 \leq i \leq n$.

Conjecture 52 : Let G be a simple $2d$-regular graph on at most $4d + 1$ vertices where $d \geq 1$. Then G admits a hamiltonian decomposition provided G is of odd order. (Nash-Williams, 1971)

Conjecture 53 : Let $G \square H$ be the cartesian product simple graphs G and H. Then $G \square H$ admits a hamiltonian decomposition provided that G and H admit hamiltonian decompositions. (Bermond, 1978)

Problem 54 : If G is a simple hamiltonian 3-regular graph, does the prism over G, $G \square K_2$, admit a hamiltonian decomposition?

> **Best result :** No counterexamples have been found when G is 2-connected.

Theorem 55.1 : Let G be a 4-regular graph, and let e and f be two edges of G. Then the number of hamiltonian decompositions of G in which e and f belong to different hamiltonian cycles is even. (Thomassen, 1978)

Theorem 55.2 : The line graph of a simple 3-regular graph admits a hamiltonian decomposition if and only if the graph is of class 1, i.e., 3-edge-colorable.

Theorem 55.3 : The line graph of a simple 4-regular graph admits a hamiltonian decomposition if the graph itself admits a hamiltonian decomposition.

Conjecture 55 : The line graph of a simple $2d$-regular graph admits a hamiltonian decomposition if and only if it has edge connectivity $4d-2$. (Jackson, 1991) (Note : Some 3-connected 4-regular graphs are not hamiltonian.)

Conjecture 56 : Every diregular bipartite tournament admits a decomposition into directed hamiltonian cycles. (Jackson, 1981)

Definition : (Cycle Double Cover) A cycle double cover is a collection of cycles such that each edge of the graph belongs to exactly two cycles of the collection.

Conjecture 57 : (Cycle Double Cover Conjecture)

Every 2-edge-connected graph admits a cycle double cover. (Tutte, 1950s; Szekeres, 1973; Seymour, 1979; Jaeger, 1985)

Best result : (Goddyn, 1988) The girth of a minimal counterexample is 10.

Conjecture 57′ : Every 2-edge-connected graph can be embedded in some surface in such a way that each face is bounded by a cycle.

$57' \Rightarrow 57$

Conjecture 57″ : Every 2-edge-connected graph G has a cycle double cover with at most $|V(G)|-1$ cycles.

Conjecture 58 : Every 2-edge-connected graph admits a double cover by five even subgraphs. (Celmins, 1984; Preissmann, 1981)

Theorem 59.1 : (Jaeger, 1976; Kilpatrick, 1975)

(i) Every 4-edge-connected graph is the union of two even subgraphs.

(ii) Every 2-edge-connected graph is the union of three even subgraphs.

Theorem 59.2 :

(i) Every 4-edge-connected graph admits a double cover by three even subgraphs.

(ii) Every 2-edge-connected graph admits a quadruple cover by seven even subgraphs.

Conjecture 59 : Every 2-edge-connected graph adits a quadruple cover by six even subgraphs. (Jaeger, 1985)

Conjecture 60 : Every 2-edge-connected 3-regular graph admits a double cover by six matchings.

Known results :

Theorem 60.1 : (Fan, 1992)
Every 2-edge-connected graph has a 6-tuple cover by 10 even subgraphs. (For 3-regular non-3-edge-colorable graphs, 10 is least possible).

Theorem 60.2 : (Chen, 1971; Gallai, 1979)
Every graph can be decomposed into an even graph and an edge cut.

Definition : (Edge-weighted graphs) A graph G with a weighted function defined on $E(G)$. A cover in a graph G induces a weighting w of G, the weight $w(e)$ being the number of elements in the cover which contain e.

Theorem 60.3 : Let G (of order n) be a simple 2-connected edge-weighted graph induced by a cycle cover. Then G admits a cycle of weight at least $2w(G)/(n-1)$ where $w(G)$ is the total weight of G. (Bondy and Fan, 1991)

Conjecture 61 : Let $n, m_1, m_2, \ldots, m_t$ be positive integer such that n is odd $n \geq m_1 \geq m_2 \geq \ldots \geq m_t \geq 3$ and $\sum_{i=1}^{t} m_i = \binom{n+1}{2}$. Then K_n admits a decomposition of cycles with lengths $m_1, m_2, \ldots, m_t$ respectively. (Alspach, 1980)

Conjecture 61′ : If n is even, then we have a similar decomposition of $K_n - F$ where F is a 1-factor of K_n and $\sum_{i=1}^{t} m_i = \binom{n+1}{2} - \frac{n}{2}$.
(Note : Many partial results have been obtained.)

Conjecture 61″ : Let $K_{m(n)}$ be an even graph where $m \geq 3$ such that $\sum_{i=1}^{t} m_i = \binom{m}{2} \cdot n^2$. Then $K_{m(n)}$ admits a cycle decomposition with cycle lengths $m_1, m_2, \ldots, m_t$, respectively.

Conjecture 61‴ : If $m = 2$, then we only consider even m_i's and $n \neq 4$.

Conjecture 62 : Let $m_1, m_2, \ldots, m_t$ be positive integers such that $\sum_{i=1}^{t} m_i = \binom{m+1}{2}$. Then K_n can be decomposed into paths of lengths $m_1, m_2, \ldots, m_t$ respective.(Fu, 2002)

Conjecture 63 : Let G be a graph of order n and size m, $\binom{m+1}{2} \leq m < \binom{n+2}{2}$. Then $E(G)$ can be partitioned into n sets $E_1, E_2, \ldots, E_n$ such that $|E_i| < |E_{i+1}|$ and $< E_i >_G \leq < E_{i+1} >_G$ for $i = 1, 2, \ldots, n-1$. (Alvai et al. 1987; Ascending Subgraph Conjecture)

Conjecture 63′ : Let G be a graph of order n and size $\binom{n+1}{2}$. Then G can be decomposed into n subgraphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ and $G_i \leq G_{i+1}$ for $i = 1, 2, \ldots, n-1$. ($H \leq G$ denotes H is a subgraph of G.) (Alvai et al., 1987)

Conjecture 64.1 : Let $T_2, T_3, \ldots, T_n$ be trees of order $2, 3, \ldots, n$ respectively. Then K_n can be decomposed into $n-1$ subgraph $G_1, G_2, \ldots, G_{n-1}$ such that $G_i \cong T_{i+1}$ for $i = 1, 2, \ldots, n-1$. (Gy*á*rf*á*s, 1978)

Conjecture 64.2 : Every graph with n vertices and more than $(k-1)n/2$ edges contains every tree with k edges. (Erdös and S*ó*s, 1963)
(Note : $(k-1)n/2$ is the number of edges in a graph of order n which ensures the existence of a path of length k and a star with k edges.)

Conjecture 64.3 : (Weaker form)
For every $k \geq 1$ there is an $n(k)$ such that if $n \geq n(k)$ and $T_{n-k}, T_{n-k+1}, \ldots, T_n$ are trees, with T_i having i vertices, then they can be packed into K_n.

Best result : Conjecture 64.1 holds for

(i) T_i is a path or a star. (Gy*á*rf*á*s and Lehel, 1978).

(ii) T_i's are stars except two. (Gy*á*rf*á*s and Lehel, 1978).

(iii) T_i's are trees of diameter at most 3. (Hobbs, 1981)

Conjecture 65 : Let T be an arbitrary tree of order n. Then K_{2n+1} can be decomposed into copies of T, denoted by $T|K_{2n+1}$.
(Note : If T has a graceful labelling, then Conjecture 65 holds.)

Conjecture 66 : Let T be an arbitrary odd forest of order n (even) such that $\binom{n}{2} - |E(T)|$ is multiple of $k(\geq 3)$. Then $K_n - T$ can be decomposed into cycles of length k. (Fu, 2004)

Known results : The conjecture holds for $k = 4$ and 6.

Definition : (k-sufficient) A graph G is said to be k-sufficient if $k \mid |E(G)|$ and G is an even graph.

Conjecture 67 : Let graph G is said to be k-sufficient graph of order n and $\delta \geq \lceil \frac{3n}{4} \rceil$. Then G can be decomposed into k-cycles provided that n is sufficiently large. (Fu, 2004)

(Note : n can be determined by k, i.e., for fixed k, n is also fixed.)

Conjecture 67′ : Let G be a 4-sufficient graph of order n and $\delta \geq \lceil \frac{3n}{4} \rceil$. Then G has a 4-cycle decomposition.

Problem 68 : Find an infinite class of graphs of order $n \neq q^2 + q + 1$ which are extremal graphs contain no 4-cycles where q is a prime power.

Best result : If q is a prime power and $n = q^2 + q + 1$, then it is known.

Construction (Erdös and Rényi, 1962)

The vertex set of G is the set of q^2+q+1 points of the finite projective plane $PG(2, q)$ over the finite field of order q. A point is joined to all the points on its polar w.r.t. the conic $x^2 + y^2 + z^2 = 0$. Thus two points (a, b, c) and (α, β, γ) are adjacent if and only if $a\alpha + b\beta + c\gamma = 0$. Then a point not on the conic is incident to $q + 1$ points, i.e., to all the lines on its polar, while each of the $q + 1$ points on the conic is incident to q points on its polar except itself.

Problem 69 : Find $ex(n; C_4, C_5)$.

Best result : If $n = 2(q^2+q+1)$ for some prime power q, then $ex(n; C_4, C_5) \geq (q - 1)(q^2 + q + 1)$ which is obtained from variety-block bipartite graph.

Problem 69′ : Is $ex(n; C_4, C_5) = (q - 1)(q^2 + q + 1)$ for $n = 2(q^2 + q + 1)$ and q is a prime power?

Problem 69″ : Find $ex(n; C_k)$ for $n \geq k \geq 5$ and determine the extremal graphs.

Problem 70 : Find a proof of the four color theorem without using computers.

Problem 71 : Let G be the union of at most k complete graphs of order k of which any two have at most one vertex in common.
Is $\chi(G) = k$? (P. Erdös, 1981)

Problem 72 : Let $V(G) = \mathbb{R}^2$, i.e., the vertices of G are all the points in the plane, where $xy \in E(G)$ iff x and y have distance 1.
What is $\chi(G)$? (Hadwiger and Nelson)

Best results :

Theorem 72 : (De Bruijn and Erdös, 1951)
If all finite subgraphs of an infinite graph G are k-colorable, then G is k-colorable.

Proof.

Let G be a possible counterexample. By Zorn's lemma, G may be assumed to be maximal, i.e., $G+e$ gives a finite subgraph G_e of $G+e$ which is not k-colorable. Suppose $xy \notin E(G)$, $yz \notin E(G)$ but $xz \in E(G)$. Then $(G_{xy}-xy)\cup(G_{yz}-yz)+xz$ would be a finite subgraph of G is not k-colorable. Hence non-adjacency is an equivalence relation on G. Since G is not k-colorable, the number of equivalence classes is at least $k+1$, but then $K_{k+1} \subset G$ which is a contradiction. ∎

Conjecture 73 : Let G be 5-chromatic. Then G contains a subdivision of K_5.

Best results :

1. Any 4-chromatic graph contains a subdivision of K_4. (Dirac, 1952)
2. If $k \geq 7$, then the corresponding statement is not true. (Catlin, 1981)
3. The statement "G contains a subdivision of a $K_{\chi(G)}$" is false for almost all graphs. (Erdös and Fajtlowiz, 1981)

Conjecture 74 : $\chi(G) \geq k$ implies K_k is the only k-contraction critical graph.

Conjecture 75 : Let $w(G) = max_{H\leq G}\lceil |E(G)|/\lfloor \frac{1}{2}|V(H)|\rfloor \rceil$. Then $\chi'(G) = w(G)$ provided $\chi'(G) \geq \Delta(G)+2$. (Multigraph) (Andersen, 1977; Seymour, 1979)

Best result :

Theorem : (Shannon, 1949)
$\chi'(G) \leq \frac{3}{2}\Delta(G)$.

Proof :

Let $k = \chi'(G) \geq \frac{3}{2}\Delta(G)$ and G' be a k-edge-critical subgraph of G. By Vizing's Theorem, $\chi'(G) \leq \Delta(G) + \mu(G)$ where $\mu(G)$ is the multiplicity of G, there are two vertices x and y joined by at least $\lceil \frac{1}{2}\Delta(G)\rceil$ edges. Now, delete an $x-y$ edge e from G' and let φ be a $(k-1)$-edge-coloring of $G'-e$. In φ, the number of the $k-1$ colors missing at x (and y respectively) is $(k-1)-(\Delta(G)-1)$, but no color is missing at both x and y, hence $2(k-\Delta(G)) + \lceil \frac{1}{2}\Delta(G)\rceil - 1 \leq k-1$. This is implies that $k \leq \frac{3}{2}\Delta(G)$. ■

Definition 76 : (Flow) A k-flow is an assignment of a direction and an integer from $\{0, 1, 2, \ldots, k-1\}$ to each edge of a graph so that at each vertex the flow out equals the flow in. A nowhere-zero flow is a flow in which no edge receives the integer 0.

Conjecture 76 : Let D be a 2-edge-connected graph. The G has a nowhere-zero 5-flow. If G contains no subdivision of the Petersen graph, then G has a nowhere-zero 4-flow.

Theorem 77.1 : If G is cubic, then G has a nowhere-zero 4-flow if and only if G is 3-edge-colorable.

Theorem 77.2 : A graph G has a nowhere-zero 4-flow if and only if $E(G)$ is the union two even graphs.

Theorem 77.3 : Take a planar drawing of a planar graph G. The regions of this drawing may be colored with k colors, such that for each edge the two regions bordering it have different colors, iff G has a nowhere-zero k-flow.

Conjecture 77 : (Equivalent to Conjecture 76)
Let G ba a planar graph. Then G is 3-edge colorable iff

(i) every vertex of G has valency at most 3, and

(ii) no subgraph of G has one vertex of degree 2 and all the others of degree 3. (Grötzsch)

Conjecture 78 : Let G be a planar graph, and let $k \geq 0$ be an integer. Then G is k-edge-colorable iff

(i) every vertex of G has valency at most k, and

(ii) for every $X \subset V(G)$ with $|X|$ odd, there are at most $\frac{1}{2}k(|X|-1)$ edges with both ends in X.

(Seymour, 1979)

Conjecture 79 : Every 4-edge-connected graph has a nowhere-zero 3-flow.

Best result : (Jaeger, 1975) Every 4-edge-connected graph has a nowhere-zero 4-flow.

Proof :
Use the result by Nash-Williams : For any $k \geq 1$, every $2k$-edge-connected graph has k mutually edge-disjoint spanning trees, and Theorem 77.2. (Define two even graphs C_1, C_2, such that $C_1 \cup C_2 = E(G)$.) ■

Conjecture 80 : Every graph with no cut-edge (bridge or isthmus) has a nowhere-zero 5-flow.

Best result : (Seymour, 1981) Every graph with no isthmus has a nowhere-zero 6-flow. (An isthmus is an edge which is not in any cycle.)

Conjecture 81 : (Crossing number)
$cr(K_{m,n}) = \lfloor m/2 \rfloor \lfloor (m-1)/2 \rfloor \lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor$.

Best result : Conjecture holds for $min\{m,n\} \leq 6$. (Kleitman, 1970)

Conjecture 81′ : ("Conjecture 81 holds" implies "Conjecture 81′ holds".)
$cr(K_{m,n}) = \overline{cr}(K_{m,n})$ where $\bar{cr}(K_{m,n})$ is the rectilinear crossing number of $K_{m,n}$. (Eggleton, 1986)

Problem 82 : Find $cr(K_n)$.

Conjecture 82$'$: $\overline{cr}K_n > cr(K_n)$ for all $n \geq 10$.

Best results :

1. $\overline{cr}(K_8) > cr(K_8) = 18$. $\overline{cr}(K_9) = cr(K_9) = 36$. $\overline{cr}(K_{10}) > cr(K_{10})$.
2. $\overline{cr}(K_n) \geq cr(K_n) \geq \frac{3}{10}\binom{n}{4}$ for n sufficiently large. (Kleitman, 1970)
3. $lim_{n\to\infty} cr(K_n)/n^4$ exists.

Definition : (Linear Arboricity) Let H be the collection of subgraphs of a graph G which are linear forest. Then $la(G) = cov^\star(G, H)$.

Conjecture 83 : $la(G) = \lceil\frac{\Delta(G)+1}{2}\rceil$. (Akiyama et al., 1980)

Known result : $\Delta \leq 6$, $\Delta'(G) = 8$ and $\Delta(G) = 10$.

Definition : (Linear k-arboricity) The length of a path in a linear forest is at most k.

Conjecture 84 : $la_k(G) \leq \begin{cases} \lceil\Delta(G)\cdot\frac{n+1}{2}\cdot\lfloor\frac{kn}{k+1}\rfloor\rceil & \text{if } \Delta(G) \neq n-1\text{; and} \\ \lceil\Delta(G)\cdot\frac{1}{2}\cdot\lfloor\frac{kn}{k+1}\rfloor\rceil & \text{if } \Delta(G) = n-1.\ (n = |V(G)|) \end{cases}$
(Habib and Peroche, 1982)

Conjecture 85 : $cov^\star(n, \{\text{cycles}, K_3\}) \leq cn$ for some $c > 0$. ($c = \frac{3}{2}$)
(Erdös and Gallai)

Problem 86 : Is it true that every 3-edge-connected graph G can be covered by $e(G) - n + 1$ minimal cuts and edges?

Conjecture 87 : If maximal cliques are removed one by one from any n vertex graph, then the graph will be empty after at most $\frac{n^2}{4}$ steps. (Winkler, 1990)

Conjecture 88 : If maximal cliques are removed one by one from any n vertex graph, then the graph will be empty after the sum of the number of vertices reach at most $\frac{n^2}{2}$. (Winkler, 1990)

Conjecture 89 : Every bipartite graph with 2^k edges contains an induced subgraph with 2^{k-1} edges. (Chang an Hwang)

Definition : (Turán's Numbers)

1. For simple graph; $T(n, H) = max\{|E(G)| \mid H$ is not a subgraph of G and $|V(G)| = n\}$.
2. For k-uniform hypergraph; $T_k(n, H) = max\{|E(G)| \mid H$ is not a subgraph of G and $|V(G)| = n\}$.

   Known results :

Theorem 90.1 : $T(n, K_3) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

Theorem 90.2 : $T(n, K_{k+1}) \sim (1 - \frac{1}{k}\binom{n}{2})$.

Conjecture 90′ : $T(n, K_4) = \frac{5}{9}\binom{n}{3}$.

Problem 90″ : $T_3(n, K_{k+1}) =?$.

Problem 90‴ : $T(n, K_{k,k}) =?$

Best result : $cn^{2-\frac{1}{2k-1}} < T(n, K_{k,k}) < cn^{2-\frac{1}{k}}$.

Definition : H is (n, e)-unavoidable if any graph with $|V(G)| = n$ and $|E(G)| = e$ contains H. Define $f(n, e) = max\{|E(G)| \mid H$ is (n, e)- unavoidable$\}$.

Known results :

Theorem 91.1 :

$$f(n,e) = \begin{cases} 1, & \text{if } e \leq \lfloor \frac{n}{2} \rfloor \\ 2, & \text{if } \lfloor \frac{n}{2} \rfloor < e \leq n; \\ (\frac{e}{n})^2 + O(\frac{e}{n}), & \text{if } n < e \leq n^{4/3}; \text{ and} \\ \frac{\sqrt{e}\log n}{\log(\binom{n}{2}/e)}, & \text{if } n^{4/3} < e << n^2. \end{cases}$$

Problem 91 : For $t \geq 3$, find the range of e such that $f(n, e) = t$.

Problem 92 : How many edges must a graph have which contains all planar graphs on n vertices? (Fan Chung Graham)

Definition : (Graceful Labelling) A graceful labelling of a graph G is a 1-1 mapping λ from $V(G)$ into $\{0, 1, 2, \cdots, |E(G)|\}$ such that $\{|\lambda(a) - \lambda(b)| \mid ab \in E(G)\} = \{1, 2, \cdots, |E(G)|\}$.

Conjecture 93 : All trees are graceful, i.e., every tree has a graceful labelling.

Note : This is one of the most interesting problems in Graph Theory.

Conjecture 94 : Let G be a graph of order at least 3. Then G can be determined uniquely (up to isomorphism) by $\{G - v\}_{v \in V(G)}$.

Note: This is also known as the Reconstruction Conjecture.

Problem 95 : Let G_1 and G_2 be two graphs with the same order and size. Dose there exist a collection of properties $\{P_i\}_{i \in I}$ such that G_1 is isomorphic to G_2 provided they both satisfy the above properties.

Definition : (Ramsey Number) $r(k, k) = min\{n|$ any 2-coloring of the edges of K_n contains a monochromatic $K_k\}$.

Problem 96$'$: For $k \geq 5$, $r(k, k) =?$

Known results :

1. $r(3, 3) = 6$, $r(4, 4) = 18$, and $43 \leq r(5, 5) \leq 55$.
2. $\binom{2k-1}{k-1} \geq r(k, k) \geq (1 + o(1)) \frac{k}{e\sqrt{2}} 2^{k/2}$.
3. $4 \geq lim_{k \to \infty} (r(k, k))^{1/k} \geq \sqrt{2}$.

Problem 96$''$: $lim_{k \to \infty} (r(k, k))^{1/k} =?$

Problem 96$'''$: $r(k, k) - r(k - 1, k - 1) \leq$?

Definition : (Expander Graphs) Let S be a set of vertices in a graph G. Then G is an α-expander if $\forall\ S \subseteq V(G)$, $|N(S) \cup S| \geq \frac{\alpha|S|}{\frac{(\alpha-1)|S|}{n} + 1}$.

Known results :

Theorem 97.1 : If G is a random k-regular graph, then G is a k-expander. (Tanner)

Theorem 97.2 : If G is k-regular with eigenvalues $\lambda_1,\ \lambda_2,\ \cdots\ ,\ \lambda_n$ such that $\lambda_1 = k$, $|\lambda_i| \leq \lambda_2 \forall\ i \neq 1$, then G is a $\frac{k^2}{\lambda_2^2}$-expander.

Theorem 97.3 : There exist $\frac{k}{4}$ - expanders. (by constructing a k-regular graph with $\lambda_2 \leq 2\sqrt{k-1}$.) (Lubotzky et al.)

Problem 97 : Construct $(k + o(k))$-expanders.

Definition : (Spectrum of G) The multiset of the eigenvalues of G is the spectrum of G, denoted by $sepc(G)$.

Problem 98 : Construct cospectra graphs. Does there exist a property P such that if G has property P, then G is the only graph with $spec(G)$?

Conjecture 99 : Let $\chi_t(G)$ be the total chromatic number of a graph G. Then $\chi_t(G) \leq \Delta(G) + 2$.

Conjecture 100 : $P \neq NP$.

Theorem 101.1 : (Generalized Ramsey's Theorem) Given positive integers k, and a collection of k finite graphs $G_1,\ G_2,\ \cdots,\ G_k$, there is a least positive integer $r(G_1, G_2, \cdots, G_k)$ such that for any k coloring of a complete graph K_n with $n \geq r(G_1, G_2, \cdots, G_k)$, there is for some $1 \leq i \leq k$ such that all the edges of G_i are of color i. (i-monochromatic)

Note: The case when $G_i \cong K_i$ gives the general Ramsey's Theorem.

Definition : (Ramsey Number) $r(G_1, G_2, \cdots, G_k)$ is called the Ramsey number of $\{G_1, G_2, \cdots, G_k\}$.

Known results :

1. For $2 \leq m \leq n$, $r(P_m, P_n) = n + \lfloor \frac{m}{2} \rfloor - 1$. (Geréncser and Gyárfas, 1967)
2. Let $n_1, n_2, \cdots, n_k$ be positive integers not less than 2. Then there is a least positive integer $r(n_1, n_2, \cdots, n_k)$ such that for any coloring of K_p with $p \geq r(n_1, n_2, \cdots, n_k)$, there is some i a complete graph K_{n_i} all of whose edges are colored i. (Ramsey, 1930)
3. When at least one of m or n is even and T_m and T_n are stars of size m and n respectively, $r(T_m, T_n) = m + n$.

Conjecture 101 : For any trees T_m (of size m) and T_n (of size n) with $m, n \geq 1$, $r(T_m, T_n) \leq m + n$. (Burr and Erdös, 1976)

Conjecture 102 : Any graph with n vertices and at least $\frac{n(k-1)}{2} + 1$ edges contains any tree of size k, $k \geq 1$. (Erdös - Sós)

Note: The truth of Conjecture 102 implies the truth of Conjecture 101.

Conjecture 103 : For $n \geq 5$ and odd, $r(C_n, C_n, C_n) = 4n - 3$. (Bondy and Erdös, 1973)

Known results :

Theorem 104.1 : If $3 \leq m \leq n$ with $(m, n) \neq (3, 3), (4, 4)$, then

$$r(C_m, C_n) = \begin{cases} 2n - 1, & \text{when } m \text{ is odd;} \\ n + \frac{m}{2} - 1, & \text{when } m \text{ and } n \text{ are even; and} \\ max\{n + \frac{m}{2} - 1, 2m - 1\}, & \text{otherwise.} \end{cases}$$

Theorem 104.2 : $r(C_5, C_5, C_5) = 17$.

Theorem 104.3 : $r(C_7, C_7, C_7) = 25$. (Faudree et al.)

Theorem 104.4 : $r(C_n, C_n, C_n) \leq (4 + o(1))n$. (Luczak, 1999)

Conjecture 105 : If $m \geq 3$ and $n \geq m$ except for $n = m = 3$, then $r(K_m, C_n) = (m - 1)(n - 1) + 1$.

Known results :

1. Conjecture holds for $m = 3, 4, 5, 6$.
2. Conjecture holds for $n \geq 4m + 2$.

Definition : (Size Ramsey Number) Given $k \geq 2$ and graphs $G_1, G_2, \cdots, G_k$, a graph F is said to "*arrow*" the k-tuple $(G_1, G_2, \cdots, G_k)$ denoted by $F \rightarrow (G_1, G_2, \cdots, G_k)$ if for any k-coloring of F there is for some i a monochromatic copy of G_i such that each edge of G_i is colored i. The size Ramsey number $\hat{r}(G_1, G_2, \cdots, G_k)$ is the smallest size of a graph G such that $F \rightarrow$

$(G_1, G_2, \cdots, G_k)$. A graph F is $(G_1, G_2, \cdots, G_k)$-minimal if $F \to (G_1, G_2, \cdots, G_k)$, but no proper subgraph of F also arrows.

Known results :

1. $\hat{r}(K_m, K_n) = \binom{r(m,n)}{2}$.
2. $\hat{r}(K_{1,m}, K_{1,n}) = m + n - 1$.
3. If $F \to (G, H)$, then F has at least $|E(G)| + |E(H)| - 1$ edges.
4. The fact that $K_{r(G,H)} \to (G, H)$ implies that $\hat{r}(G, H) \leq \binom{r(G,H)}{2}$.
5. For any tree T of order n with maximum degree Δ, there is a constant c such that $\hat{r}(T, T) \leq c\Delta n$. (Haxell and Kohayakawa, 1995)

Conjecture 106 : For a graph G of order n and bound degree Δ, there is a constant $c = c(\Delta)$ such that $\hat{r}(G, G) \leq cn$. (Beck Conjecture, 1990)

Conjecture 107 : (Star Forest Conjecture) Let s and t be positive integers with $m_1 \geq m_2 \geq \cdots \geq m_s \geq 1$ and $n_1 \geq n_2 \geq \cdots \geq n_t \geq 1$ and let $F_1 = \bigcup_{i=1}^{s} K_{1,m_i}$ and $F_2 = \bigcup_{j=1}^{t} K_{1,n_j}$. Then $\hat{r}(F_1, F_2) = \sum_{k=2}^{s+t} P_k$ where $P_k = max\{m_i + n_j - 1 | i + j = k\}$.

Note: The restricted size Ramsey number $r^*(G, H)$ is the minimum size graph F of order $r(G, H)$ such that $F \to (G, H)$.

Problem 107′ : What is $r^*(F_1, F_2)$?

Conjecture 108 : For each integer $n > 1$, there is a coloring of the edges of the n-cube with n colors, one being black, such that the black edges together with the edges of any other color induce a Hamiltonian cycle.

Problem 109 : Does there exist a perfect 1-factorization of K_{2n}?

Note: A 1-factorization is perfect if any two 1-factors of the 1-factorization form a hamiltonian cycle.

Definition : (Earth-Moon Map) Let A and B be two planar graphs which are embedded in the earth and the moon respectively. An earth-moon map is a pair (A, B) such that each simply connected region of A on the earth (is a "country") has a corresponding simply connected "colony" of B on the moon.

Definition : (Earth-Moon Coloring) An earth-moon coloring is a coloring of the vertices of $A \cup B$ such that (i) each country receives the same color as its colony and (ii) if two countries or colonies share a common boundary, then they receive distinct colors.

Problem 110 : What is the minimum number of colors necessary to color all earth-moon maps ? ($\chi_{em}(A, B) = ?$)

Known results : $9 \leq \chi_{em} \leq 12$. (Beineke and Harary, Sulanke)

Problem 110′ : Let G be a graph of thickness 2. Then $\chi(G) = ?$.

Note: $\chi_{em} = max\{\chi(G)|G \text{ is of thickness } 2\}$.

Problem 110″ : Let G be any graph of thickness t. Find $\chi^{(t)} = max\{\chi(G)\}$. ($\chi^{(t)} \geq 6t - 2$.; Hutchinson, 1993)

Conjecture 111 : Every bridgeless cubic graph can be edge-colored by the edges of the Petersen graph P such that three mutually adjacent edges in G receive colors that are mutually adjacent in P. (Use 15 colors.) (Jaejer, 1988)

Conjecture 112 : Every bridgeless cubic graph can be edge-colored by points of a Steiner triple system (S, t) such that three mutually adjacent edges in G form a triple in t.

Known results : The graphs with smaller genus can be colored by using Steiner triple system of order 7. (Fu, 2001)

Conjecture 113 : Let G be a bridgeless cubic graph and let H be a group of order at least 5. Then the edges of G can be colored with the non-identity elements of H such that each vertex of G the three distinct colors sum to the identity in H. (Archdeacon, 1986)

Problem 114 : Find the crossing number of Q_n (n-cube).

Known results :

1. Conjecture (Erdös and Guy) $\lim_{n\to\infty} cr(Q_n/n^4) = 5/32$.
2. $cr(Q_n) \leq 4^n/6 + o(1)$. (Madej, 1991)

3. $cr(Q_n) \geq 4^n/20 + o(1)$. (Sýkora and Vrťo, 1993)

4. $cr(Q_4) = cr(C_4 \times C_4) = 8$.

Conjecture 115 : $cr(C_m \times C_m) = n(m-2)$ where $m \leq n$.

Known results :

1. $n \leq 5$ is known. (Stobert, 1993)
2. $cr(C_3 \times C_n)$ is known. (Ringeisen and Beineke, JCB(B), 1978).

Definition : (Polyhedral Graphs) A finite graph G is called polyhedral iff it is isomorphic to the graph of vertices and edges of a 3-dimensional covex polyhedron.

Theorem : (E. Steinitz) Polyhedral graphs coincide with 3-connected planar graphs.

Problem 116 : Find the maximal and minimum length of a simple cycle in a polyhedral graph with v vertices.

Conjecture 116′ : There exists a positive constant c such that every polyhedral graph with v vertices has a cycle of length at least cv^s with $s = \log 2/\log 3$.

Conjecture 117 : Every bridgeless cubic graph has a collection of six perfect matchings together contain every edge exactly twice. (Fulkerson's Conjecture, 1971)

Note: Almost all graphs satisfy the conjecture.

Problem 118 : Given a 3-regular graph G and a non-negative integer k, is it NP-complete to determine if there is a drawing of G in the plane with exactly k crossings?

Problem 118′ : Given a 3-regular graph G and non-negative integer k, is it NP-complete to determine if G embeds in the sphere with k handles?

Note: D. Archdeacon conjectures that both of them are NP-complete.

Definition : (Book Embedding) A page is a closed half-plane. A book is a collection of pages identified along the boundary of the half-planes. This common boundary is called the spine. A book embedding is a drawing such that all vertices lie on the spine and no edge contains a vertex on the spine other that its ends.

Definition : (Page Number) The page number of a graph G, $pn(G)$, is the fewest number of pages in a book embedding of G.

Problem 119 : Find $pn(K_{n,m})$.

Best result : $pn(K_{n,m}) \leq \lceil (2n+m)/4 \rceil$ and equality holds when $n = m$. (Muder et al., JGT, 1988)

Conjecture 120 : Let $r_{avg}(G)$ denote the expected number of regions in a randomly selected orientable embedding of the graph G. Then $r_{avg}(K_n) = 2\ln n + O(1)$.

Best result : For each $\epsilon > 0$ there exists a real number $b(\epsilon)$ such that $r_{avg}(K_n) \leq (1+\epsilon)(n-1) + b(\epsilon)$. (S. Stahl, JGT 1995)

Conjecture 121 : A random graph almost always has an edge partition into K_4's and K_5's. (D. Archdeacon)

Conjecture 122 : Every planar graph can be edge-partitioned into two outerplanar graphs. (Chartrand et al., JCT(B), 1971)

Definition : (Pebbling Move) A pebbling of a graph G is a mapping $\varphi : V(G) \to N \cup \{0\}$. A pebbling move is of a pebbling is to move two pebbles from one vertex u to its adjacent vertex v and obtain one pebble on that vertex, i.e., after the move we have a new pebbling ψ such that $\psi(u) = \varphi(u) - 2$ and $\psi(v) = \varphi(u) + 1$.

Definition : (The Pebbling Number) The pebbling number of a graph G, denoted by $\alpha(G)$, is the minimum number of pebbles which can be arbitrarily assigned to the vertices of G, $\sum_{v \in V(G)} \varphi(v)$, such that for each vertex w, there exists a sequence of pebbling moves $\psi_1, \psi_2, \cdots, \psi_t$, of G satisfy $\psi_t(w) \geq 1$.

Facts :

1. $\alpha(G) \geq 2^d$ where d is the diameter of G.
2. $\alpha(G) \geq |V(G)|$.

Conjecture 123 : $\alpha(G \times H) = \alpha(G)\alpha(H)$.

Definition : (The Optimal Pebbling Number) The optimal pebbling number of a graph G, denoted by $\alpha'(G)$, is the minimum number of pebbles which can be assigned (in an optimal way) to the vertices of G such that each vertex can obtain a pebble by a sequence of pebbling moves.

For Example : $\alpha(C_4) = 4$ and $\alpha'(C_4) = 3$.

Problem 124 : Find $\alpha'(Q_n)$.

Problem 124′ : Let T be a tree. Find $\alpha'(T)$.

Known results : (Fu and Chin-Lin Shiue)

1. If T is a caterpillar, then $\alpha'(T)$ is determined.
2. If T is a complete t-ary tree, then $\alpha'(T)$ is determined.

Definition : (Prime Labeling) A prime labeling φ of a graph G is a 1-1 mapping from $V(G)$ onto $\{1, 2, \cdots, |V(G)|\}$ such that adjacent vertices receive labels which are relatively prime.

Conjecture 125 : Every tree has a prime labeling.

Known results : Every tree of order not greater than 533 has a prime labeling. (Fu and Kuo)