Combin. Des, Lecture 6 April 13, 14.

Date (Extremal Set Theory) No. 1

Under a constraint or a collection of constraints find the maximum number of sets satisfying the given constraint(s).

Notations

(*) Clearly, the collection of sets, $\mathbb{B}$, from X, is a design $(X, \mathbb{B})$.

1. $[n] = \{1, 2, \cdots, n\}$.

2. $\binom{[n]}{k} =_{def}$ the collection of k-subsets (all) of $[n]$.

3. $\binom{n}{k} = \left| \binom{[n]}{k} \right|$.

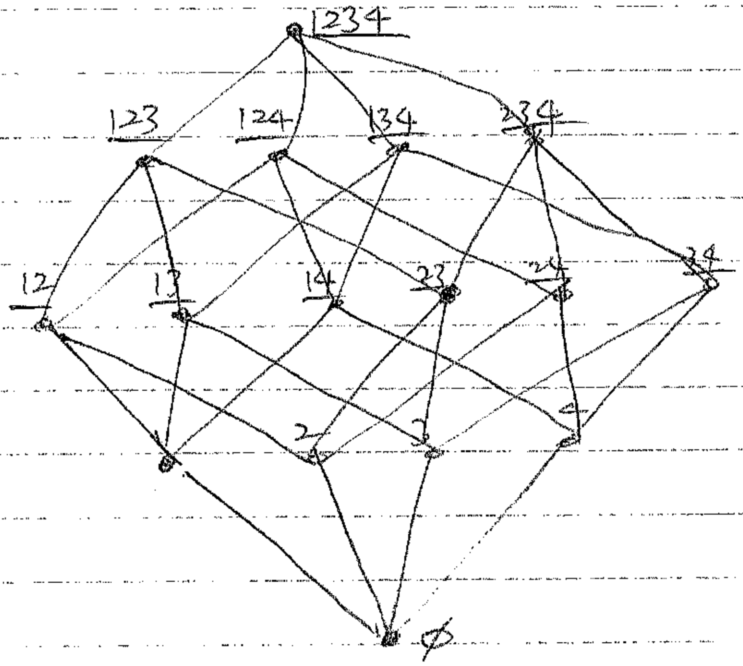
4. $X = \{x_1, x_2, \cdots, x_n\}$ is a set of n elements and "$\le$" is a partial order defined on X. $\langle X, \le \rangle$ is called a partial ordered set, Poset in short.

(o) "$\le$" is a partial order of X, if (i) $a \le a$ $\forall a \in X$, (Reflexivity) (ii) $a \le b$ and $b \le a$ implies that $a = b$, $\forall a, b \in X$, (Anti-symmetry) and (iii) $a \le b$, $b \le c$, implies that $a \le c$ $\forall a, b, c \in X$. (Transitivity)

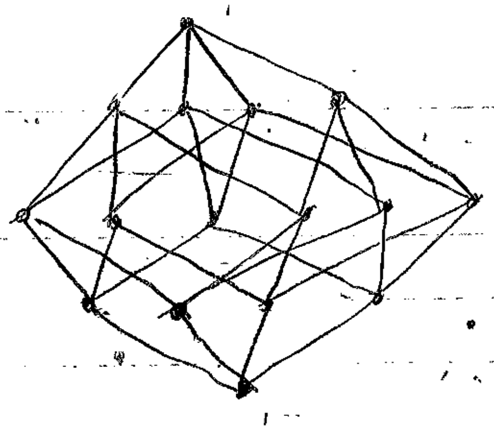
(oo) "$\le$" is a total order of Y provided any two distinct elements in Y, y_i and y_j, either $y_i \le y_j$ or $y_j \le y_i$ but not both (y_i and y_j are comparable.)

We may use a graph to depict a partial ordered set (Poset), $\langle S, \le \rangle$. It is known as the Hasse-diagram. Mainly if $a, b \in S$ and $a \le b$, then the vertex representing b is higher than a as shown in the following:

So, for example, $\langle 2^{\{1,2,3,4\}}, \subseteq \rangle$ can be represented as follows.



For convenience, this diagram can be considered as a graph:

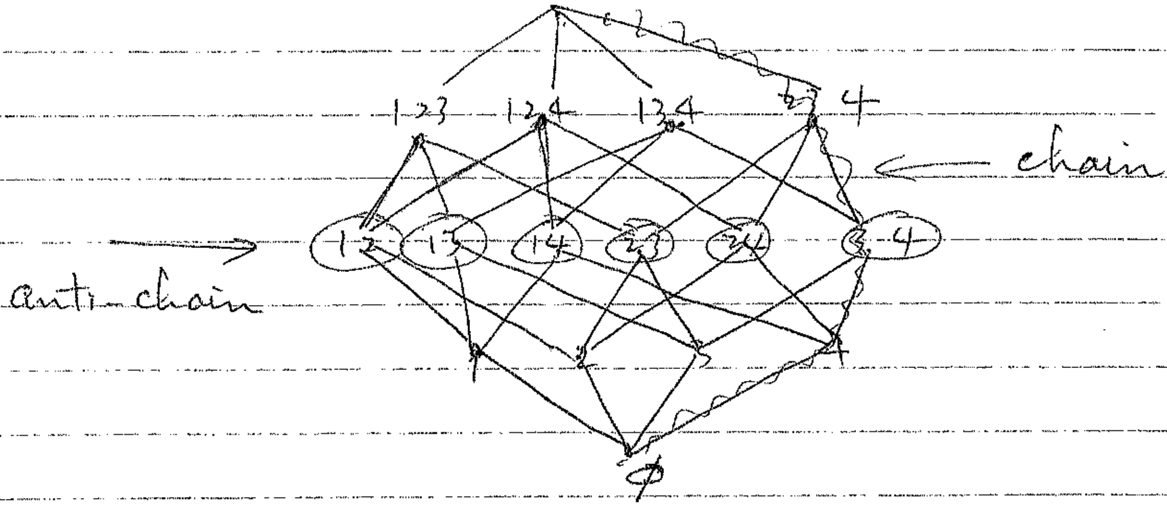


(*) Only structure will be studied.

(*) A subset of a poset in which no two distinct are comparable is called an anti-chain. On the other hand, a totally ordered set is called a chain.

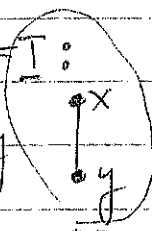
Example (Poset with set-containment)

$\{1,2,3,4\} \longrightarrow 1234$



Forbidden poset problem

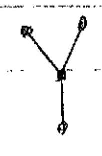
$P_2 = I$:



Given a configuration of posets, say ($y \le x$), find the maximum number of sets in $2^{[n]}$ such that the induced partial ordered set contains no sub-poset (I) which is given.

(**) We can change I ($= P_2$) to ... . For example, P_3:

or



(Y or S_3). Star

The result solving P_2 case is known as the Sperner's Theorem.

Sperner's Theorem

Consider the collection of all subsets of $[n]$. The maximum number of subsets which do not contain each other is equal to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. (The maximum anti-chain problem.)

Proof. Let A be the collection (a) of subsets which attains the maximum. Furthermore, let a_k be the number of sets in A whose size is k. Hence, $|A| = \sum_{k=0}^{n} a_k$. (Note that a_i's may be zero.) Since $\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$ is clearly an anti-chain, $|A| \geq \binom{n}{\lfloor \frac{n}{2} \rfloor}$. So, it suffices to prove $|A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Claim (LYM inequality) $\sum_{k=0}^{n} a_k / \binom{n}{k} \leq 1$.
Lubell-Yamamoto-Mashalkin

Consider the set of permutations of $[n]$. Clearly, there are $n!$ permutations. Now, for each set S in A, we associate this set with $|S|!(n-|S|)!$ permutations by taking $\left(\begin{array}{c|c} S & [n]\setminus S \\ \alpha(S) & \alpha([n]\setminus S) \end{array}\right) = \alpha$.

(The first $|S|$ elements of this permutation are exactly S.)

Note that each permutation can only be associated with a single set in A. (?)

$$\sum_{S\in A} |S|!(n-|S|)! = \sum_{k=0}^{n} a_k \cdot k!(n-k)! \le n!$$

This implies that $\sum_{k=0}^{n} a_k \cdot \frac{k!(n-k)!}{n!} \le 1$ and the proof follows.

Now, since $1 \ge \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \ge \sum_{k=0}^{n} \frac{a_k}{\binom{n}{\lfloor \frac{n}{2}\rfloor}}$, $\binom{n}{\lfloor \frac{n}{2}\rfloor} \ge \sum_{k=0}^{n} a_k = |A|$. $\blacksquare$

If it associates with two sets S_1 and S_2, then either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$.

<u>Example</u> $n=6$

$\{1,2,3\}$

(1 2 3) | (4 5 6)
(1 2)(3) | (4 5 6)
(1 3)(2) | (4 5 6)
(1 3 2) | (4 5 6)
(2 3)(1) | (4 5 6)
(1)(3)(2) | (4 5 6)
⋮

$\{1,3,4\}$

(1 3 4) | (2 5 6)
(1 4)(3) | (2 5 6)
(1 3)(4) | (2 5 6)
(3 4)(1) | (2 5 6) $\longleftrightarrow \{3,4\}$
(1 4 3) | (2 5 6)
(1)(4)(3) | (2 5 6
⋮

<u>Exercise 2-6</u> (3 points)

Find the maximum number of subsets in $2^{[n]}$ such that their induced poset does not contain P_3.

A good guess: $\binom{n}{\lfloor \frac{n}{2}\rfloor} + \binom{n}{\lfloor \frac{n}{2}\rfloor + 1}$ (?).

Another beautiful result is the maximum collection of sets $B_{n,r}$ of size r which are mutually intersecting, i.e., $\forall S_1, S_2 \in B_{n,r}$ (distinct), $S_1 \cap S_2 \neq \phi$. $B_{n,r}$ is called an r-uniform intersecting family defined on $[n]$.

<u>Theorem</u> $|B_{n,r}| = \binom{n-1}{r-1}$ $\forall n \in \mathbb{N}$. (EKR Theorem)

<u>Proof.</u> Let $B = \{ S \cup \{n\} \mid S \in \binom{[n-1]}{r-1} \}$. Then, B is an intersecting family of $[n]$ since each set contains the element n. Hence, $|B_{n,r}| \geq \binom{n-1}{r-1}$. Next, we prove that $|B_{n,r}| \leq \binom{n-1}{r-1}$.

Observe that if we let $(a_1, a_2, \cdots, a_n)$ be a cyclic permutation of $[n]$, then this cycle contains at most r sets of $B_{n,r}$. For example, $n=8$ and $r=3$, let $(3,1,8,2,7,5,6,4)$ be an arbitrary cyclic permutation. Now, if $\{8,2,7\} \in B_{8,3}$, then we have two more possible sets $\{1,8,2\}$ and $\{2,7,5\}$. So, for general n, we have at most $r \cdot (n-1)!$ sets for intersecting family. By the same idea in Sperner's Theorem, each set in $B_{n,r}$ can be associate with $r!(n-r)!$ permutations. So, $|B_{n,r}| \cdot r!(n-r)! \leq r \cdot (n-1)!$, and thus $|B_{n,r}| \leq \frac{(n-1)!}{(r-1)!(n-r)!} = \binom{n-1}{r-1}$. ∎

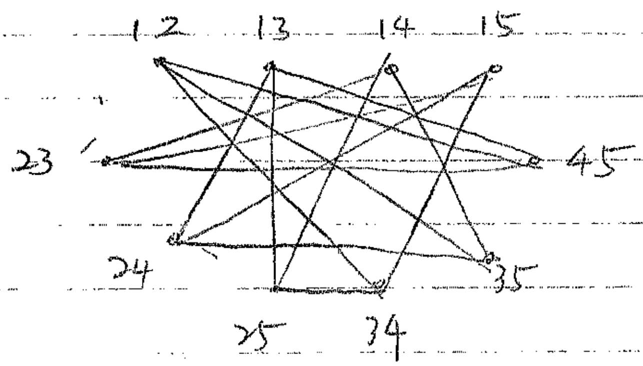
Example $|B_{7,3}| = \binom{6}{2} = 15.$

Another good problem to study related to sets.

Let $n = 2t+1$. Then, we may define a graph G as follows:

$V(G) = \binom{[n]}{t}$ and two vertices are adjacent if and only if their intersection is an empty set.

Example $n=5$, $t=2$



This graph is in fact the Petersen graph.

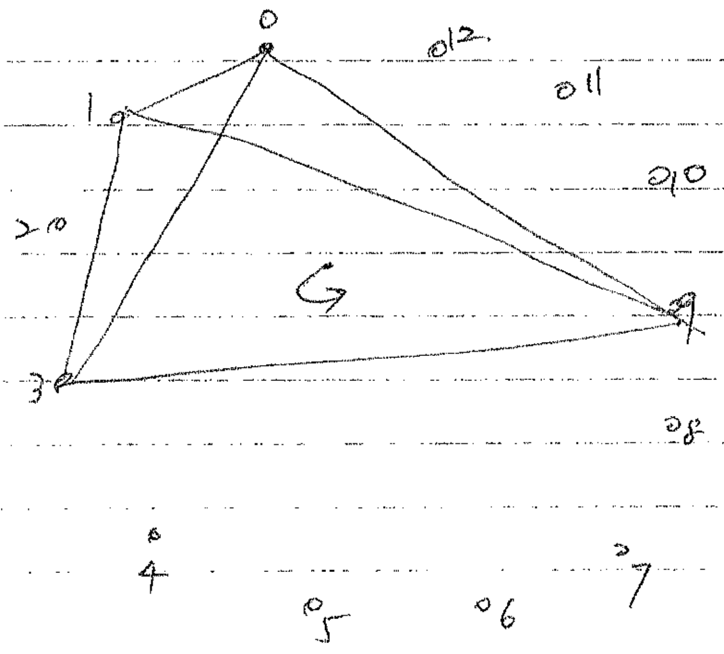
(∘) The graph G is known as an odd graph of order n, denoted by O_n

(∘∘) Study the structure of O_n is an important problem in both Graph Theory and Design Theory.

If we further require that any two r-sets in $2^{[n]}$ can have at most one element in common, thus exactly one element in common, then the collection of such sets (r-sets), denoted by $B^{(1)}_{n,r}$ has at most $\frac{n(n-1)}{r\cdot(r-1)}$ sets.

To see this, we notice that any pair of elements in $[n]$ can occur in at most one r-set of $B^{(1)}_{n,r}$. Hence, the pairs we have in total is $n(n-1)/2 = \binom{n}{2}$ and each r-set can use $\binom{r}{2} = \frac{r(r-1)}{2}$ pairs, this implies that $|B^{(1)}_{n,r}| \le \binom{n}{2} / \binom{r}{2}$.

In fact, for some n and r, the equality does hold. For example, $B^{(1)}_{7,3} = \{124, 235, 346, 457, 561, 672, 713\}$ (Fano plane) and $|B^{(1)}_{13,4}| = \frac{13\cdot 12}{4\cdot 3} = 13$. $B^{(1)}_{13,4} = \{\{0,1,3,9\}+i \mid i \in \mathbb{Z}_{13}\}$



We can check any two of them have one element in common!

Exercise 2-7 (3 points)

Give five nontrivial examples for $n, r \geq 3$, such that

$|B_{n,r}^{(1)}| = \binom{n}{2} / \binom{r}{2}$.

It is also interesting to know the family that $\bar{B}_{n,r}^{(1)}$ any two r-sets in

$\bar{B}_{n,r}^{(1)}$ intersect at one point, i.e, have a common element. The

examples mentioned in p. 8, $B_{7,3}^{(1)}$ and $B_{13,4}^{(1)}$ do have such

property.

$B_{13,4}^{(1)}$:

0	1	3	9
1	2	4	10
2	3	5	11
3	4	6	12
4	5	7	0
5	6	8	1
6	7	9	2
7	8	10	3
8	9	11	4
9	10	12	5
10	11	0	6
11	12	1	7
12	0	2	8

Exercise 2-8 (3 points)

Find $\bar{B}_{31,5}$.