L. 3. Combin. Des.
March 16–17

Sub-squares

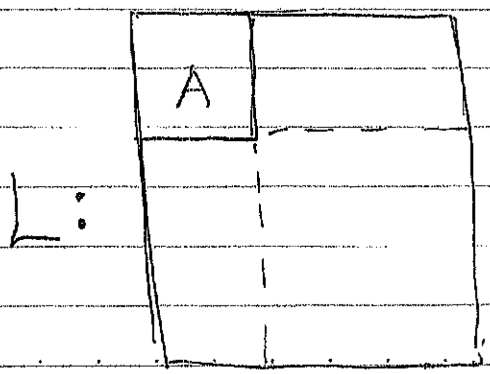
Just like algebraic structures, we have sub-quasigroup, and thus subsquares.

Definition (sub-Latin square)

If $Q' \subseteq Q$, $\langle Q', \circ \rangle$ and $\langle Q, \circ \rangle$ are quasigroups, then $\langle Q', \circ \rangle$ is called a sub-quasigroup of $\langle Q, \circ \rangle$. Their corresponding Latin squares are " Latin square and Latin subsquare respectively.

Definition (Embedding)

If A is a sub-Latin square (or Latin subsquare) of L, then A is said to be embedded in L. The standard form (of embedding) is the one with A in the upper left hand corner.



<u>Theorem 1</u> A Latin subsquare of order m can be embedded in a Latin square of order n if and only if $n \geq 2m$.

<u>Fact 2</u> If L (of order n) has a Latin subsquare A (of order m), then n may not be a multiple of m. (It is true $m \mid n$ if both L (and A) are corresponding to a $\boxed{\text{group}}$ respective.

<u>Proof of Theorem 1</u> ← <u>Exercise 1-6 (3 points)</u>

In what follows, we provide some more insight about having a subsquare.

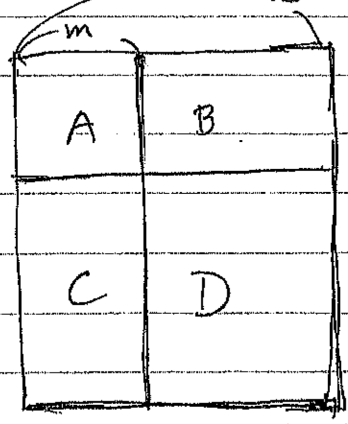
<u>Proposition 2</u> If A is embedded in L and $L(i)$ denotes the element i occurs in L (respectively A, B, C, D), then $A(i) \geq 2m-n$ where A is a Latin square of order m and L is a Latin square of order n.

<u>Proof.</u> $\forall i \in \mathbb{Z}_n$, Since $B(i) + D(i) = n-m$, $B(i) \leq n-m$.

$A(i) + B(i) = m$, Hence $A(i) = m - B(i)$

$\geq m-(n-m) = 2m-n$. ∎

L:



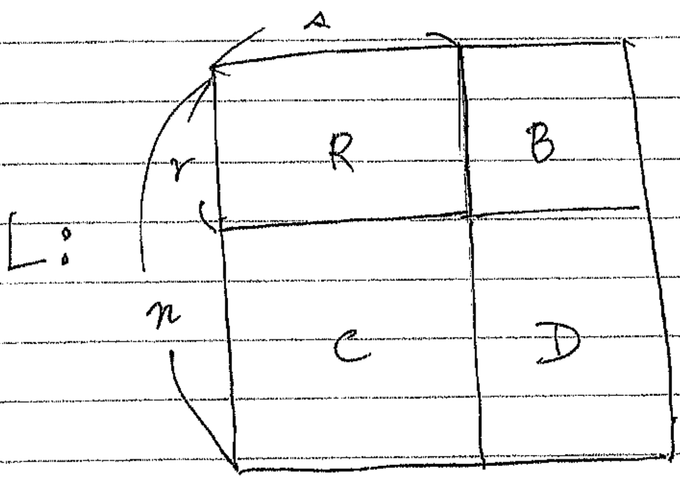
<u>Corollary 3</u> ($\Rightarrow$) of Theorem 1 is true.

<u>Proof.</u> If $n < 2m$, then every $i \in \mathbb{Z}_n$ has to occur in A which is not possible. (since A is an L.S. of order m) ∎

In fact, the subsquare A we consider here can be replaced by Latin rectangle or partial Latin rectangle. 不一定要是小的拉丁方陣。

<u>Definition</u> (Latin rectangle, partial Latin rectangle)

A Latin $r \times s$ rectangle R based on S is an $r \times s$ array such that each entry of R is filled with an element of S with an extra property: each element of S occurs in each row and each column at most once. If not every cell is filled with an entry and the property holds, then we have a partial Latin rectangle.



R is embedded in a Latin square L.

If R is embedded in a Latin square L (based on S), then

<u>Proposition 4</u>. $\forall\, i \in S,\ R(i) \geq r + s - n$.

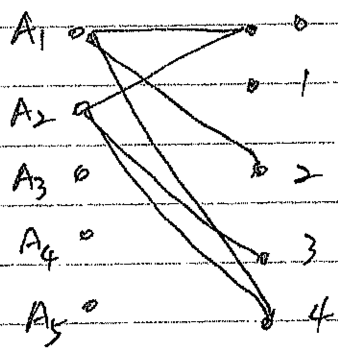
<u>Proof</u>. $R(i) + B(i) = r,\ B(i) + D(i) = n - s,\ B(i) \leq n - s. \Rightarrow R(i) = r - B(i) \geq r - (n - s).$

Proposition 5 Let R be an $r\times n$ Latin rectangle based on an n-set S. Then R can be extended (embedded) to a Latin square of order n.

Proof. Use SDR or König's Theorem. ■

0	1	2	3	4
3	2	4	1	0
A_1	A_2	A_3	A_4	A_5

$A_1 = \{0,2,4\}$, $A_2 = \{0,3,4\}$, $A_3 = \{0,1,3\}$, $A_4 = \{0,2,4\}$, $A_5 = \{1,2,3\}$



← Regular Bipartite Graph

Fact 3. Let R be an $r\times s$ partial Latin rectangle. Then R can be embedded in a Latin square of order n based on S if and only if $R(i) \geq r+s-n$, $\forall\, i \in S$ $(|S|=n)$.

(Outline)
Proof. Step 1. Fill all the entries in R, such that the condition $R(i) \geq r+s-n$ holds.

<u>Step 2.</u> Fill in the entries in B. (Obtain an $r \times n$ Latin rectangle.)

<u>Step 3.</u> Complete the Latin square by extending the rectangle.

加入 5-1, 5-2, 5-3 (侯齐孝)

<u>Definition</u> (Partial Latin Square of order n) PLS(n)

A PLS(n) is an $n \times n$ array such that each cell is either filled with an entry from an n-set S or empty, moreover, each element in S occurs at most once in each row and resp once in each column.

<u>Definition</u> (Complete the PLS(n))

Let L' be a PLS(n). L' is said to be completable if we can fill all the empty cells such that the $n \times n$ array is a Latin square.

0	1	
		2

incompletable

0	1	
1		

completable.

<u>Theorem</u> (Hoffman and H. W. Kuhn (1956)).

Let M be a given set. A necessary and sufficient condition for the sets $S_1, S_2, \cdots, S_n$ to have an SDR which includes all the elements of M is that, for every $M' \subseteq M$, at least $|M'|$ of the sets $S_1, S_2, \cdots, S_n$ have non-empty intersection with M'.

(Note that $\{S_1, S_2, \cdots, S_n\}$ has an SDR itself.)

<u>Theorem</u> (P. Hall, 1935)

$\{S_1, S_2, \cdots, S_n\}$ has an SDR if and only if the union of any k of them contains at least k elements.

<u>Proof.</u> ($\Rightarrow$) Trivial.

($\Leftarrow$) By induction on n and it's clearly true for $n=1$.

Assume that the assertion is true for all $1 \le m < n$ and consider $\{S_1, S_2, \cdots, S_n\}$.

<u>Case 1</u> $\forall\ k \le n-1 \quad \left|\bigcup_{j=1}^{k} S_{i_j}\right| \ge k+1$

Let $x_n \in S_n$ and consider $\{S_1 \setminus \{x\}, S_2 \setminus \{x\}, \cdots, S_{n-1} \setminus \{x\}\}$. By induction

SDR

<u>Case 2</u> $\exists\ h \le n-1,\ \left|\bigcup_{j=1}^{h} S_{i_j}\right| = h.$

Let $\bigcup_{j=1}^{h} S_{i_j} = \tilde{S}$. For convenience, let those h subsets be $S_1, S_2, \cdots, S_h$.

Now, consider $S_{h+1}\setminus\tilde{S}, S_{h+2}\setminus\tilde{S}, \cdots, S_n\setminus\tilde{S}$. The union of any k of these sets must contain at least k elements. For otherwise, the union of $\tilde{S}$ with these sets will contain less than $h+k$ elements, a contradiction to the assumption. Hence, $\{S_{h+1}\setminus\tilde{S}, \cdots, S_n\setminus\tilde{S}\}$ has an SDR. Also, $\{S_1, S_2, \cdots, S_h\}$ has an SDR. Together, we have an SDR for $\{S_1, S_2, \cdots, S_n\}$.

<u>Bonus problem</u>:

Prove the Theorem by Hoffman and Kuhn and then the details of H. J. Ryser's Theorem.

We can use this theorem to prove the embedding theorem proved by Ryser in 1951.

<u>Theorem</u> (H. J. Ryser (1951))

Let R be an $r \times s$ latin rectangle based on $S = \{1, 2, \cdots, n\}$ (filled partial L.S.). Then, R can be embedded in a L.S. of order n if and only if $R(i) \geq r + s - n$ for all $i \in S$.

<u>Proof</u>. The proof of necessity has been done earlier, we prove the sufficiency in what follows. We claim that R can be enlarged to an $r \times (s+1)$ rectangle R^* such that $R^*(i) \geq r + (s+1) - n$ for all i. Iteration then extends R to an $r \times n$ Latin rectangle Q based on S. Hence, R can be embedded in a Latin square of order n by applying M. Hall's extension theorem.

Let S_j denote the set of elements in S which do not occur in the jth row of R. Now, let M be the set of elements in R occurred exactly $r+s-n$ times. ($|M| \leq r$?) The proof follows
See 5-4
by showing $\{S_1, S_2, \cdots, S_r\}$ has an SDR such that the set of elements in S still satisfy the N.C.. ∎

If there are more elements which occur exactly $r-s+n$ times, say $r+1$, then the other $n-(r+1)$ elements will occur at most r times.

Hence, in total we have $(r+1)(r+s-n)+(n-r-1)\cdot r =$

$\cancel{r^2}+rs+\cancel{r}+s-\cancel{rn}-n+\cancel{nr}-\cancel{r^2}-\cancel{r} = rs+s-n < rs.$

(We will not do anything if $s=n$.)

So, by Hoffman and Kuhn's theorem, we can find an SDR such that all elements in M occur.

(*) It is interesting to know whether a PLS(n) can be completed to a Latin square.

<u>Fact 4</u> A PLS(n) with at most $n-1$ filled cells can be completed to a Latin square of order n. (Evans' Conjecture)

(In fact, the proof of this fact is quite difficult.)

Proved by B. Smetaniuk (1981). You may refer to "A course in combinatorics" by J.H van Lint and R.M. Wilson, page 189–193.

<u>Fact 5</u> It takes about 50 pages to characterize a PLS(n) with at most $n+1$ filled cells which is completable.

(L.D. Anderson and A.J.W. Hilton, 1983, LMS.)

Constructions of Latin squares with many subsquares

First, we consider the operation of two Latin squares.

<u>Definition</u> (Direct product)

Let A and B be two Latin squares based on $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. Then, the direct product of A and B, denoted by $A \otimes B$ is a Latin square of order mn based on $\mathbb{Z}_m \times \mathbb{Z}_n$ such that the entry $A_{ij} = x$ is replaced by (x, B) where (x, B) is a Latin square of order n where the (i', j') entry is filled by $(x, B_{i'j'})$.

e.g.

$$A : \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array}, \quad B : \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline \end{array}, \quad A \otimes B : \begin{array}{|c|c|c|c|c|c|} \hline (0,0) & (0,1) & (0,2) & (1,0) & (1,1) & (1,2) \\ \hline (0,2) & (0,0) & (0,1) & (1,2) & (1,0) & (1,1) \\ \hline (0,1) & (0,2) & (0,0) & (1,1) & (1,2) & (1,0) \\ \hline (1,0) & (1,1) & (1,2) & (0,0) & (0,1) & (0,2) \\ \hline (1,2) & (1,0) & (1,1) & (0,2) & (0,0) & (0,1) \\ \hline (1,1) & (1,2) & (1,0) & (0,1) & (0,2) & (0,0) \\ \hline \end{array}$$

$B \otimes A$

(0,0)	(0,1)	(1,0)	(1,1)	(2,0)	(2,1)
(0,1)	(0,0)	(1,1)	(1,0)	(2,1)	(2,0)
(2,0)	(2,1)	(0,0)	(0,1)	(1,0)	(1,1)
(2,1)	(2,0)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,1)	(2,0)	(2,1)	(0,0)	(0,1)
(1,1)	(1,0)	(2,1)	(2,0)	(0,1)	(0,0)

$B' =$

0	2	1
2	1	0
1	0	2

$B' \otimes A$

(0,0)	(0,1)				
(0,1)	(0,0)				
		(1,0)	(1,1)		
		(1,1)	(1,0)		
				(2,0)	(2,1)
				(2,1)	(2,0)

Latin squares defined on three disjoint sets.

$B' \otimes A$ is referred to as a Latin square with <u>2x2</u> holes.

(*) Let $n = h_1 + h_2 + \cdots + h_t$. If L is a Latin square of order n with t subsquares (as above) of order $h_1, h_2, \cdots, h_t$ resp. Then, L is a Latin square with holes of type $h_1 \times h_2 \times \cdots \times h_t$.

Exercise 1-7. Construct a Latin square L of order 12, such that L is commutative and also with holes of type 2^6.

(3 points)

Note. If m is odd, then L can be constructed by using direct product. But, for even m, it takes some effort!

1	2	8	5	4	7	6	3
2	1	6	7	8	3	4	5
8	6	4	3	7	2	5	1
5	7	3	4	1	8	2	6
4	8	7	1	6	5	3	2
7	3	2	8	5	6	1	4
6	4	5	2	3	1	8	7
3	5	1	6	2	4	7	8

An example, $m = 4$.