1.1 組合設計 (Combinatorial Designs)

2020, 3, 2 ~ 3, 3.

(∘) Given a finite set $X \neq \phi$.

(∘) A design (combinatorial) is an ordered pair $(X, \mathbb{B})$ where $\mathbb{B}$ is a collection of (multi)-subsets of X.

Note. Since an empty set plays no role in a design, we assume that all subsets are non-empty.

e.g. $X_1 = \{0,1,2,3\}$, $\mathbb{B}_1 = \{\{0,1,2\},\{0,1,3\},\{0,2,3\},\{1,2,3\}\}$.

$X_2 = \{0,1,2,3,4,5,6\}$, $\mathbb{B}_2 = \{\{0,1,3\}+i \mid i \in \mathbb{Z}_7\}$.
(mod 7)

(∘) For convenience, we use $\mathbb{Z}_4$ for X_1 and $\mathbb{Z}_7$ for X_2.

(*) 組合設計研究的目標在於找到滿足給定條件後如何找到 $(X, \mathbb{B})$. X_1 中任兩個元素都出現在兩個不同的集合中 ($\mathbb{B}_1$), 而 X_2 中任兩個元素都出現在 $\mathbb{B}_2$ 中唯一的一個集合中.

(∘) 考慮任兩(t)個元素出現的子集合中的設計, 簡稱為 2-design. (t-design)
...

背景 設計最早是用在實驗(或解決問題), 統計學中一般稱為實驗設計 (Experimental Design).

(*) 探討實驗設計中所隱藏的數學 (Combinatorics), 是組合設計的研究對象。

Check WiKi

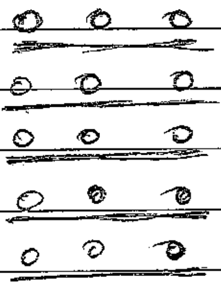
(**) 組合設計的應用非常廣泛! 魔方陣的研究 (Magic square)

歷史背景 (外國文獻), [本國文獻更早一些, 但是沒有期刊可以發表!]

最早提供和組合設計有相關聯的問題是:

Kirkman's 15 girls problem (1840左右)

15個女孩每天排路隊上學, 三個一列共5列; 每天一個女孩會認識同一列的另兩個女孩。是否可以在七天中安排好不同路隊, 讓每個女孩都認識其他的十四位呢? (如下圖)



$\boxed{\text{(Exercise 1-1) (3 points)}}$ 自己試看!

(◦) 如果是九個女孩排三列, 以下是一個答案:

0 1 2	0 3 6	0 4 8	0 5 7
3 4 5	1 4 7	1 5 6	1 3 8
6 7 8	2 5 8	2 3 7	2 4 6

對應於 $(X_3, \mathbb{B}_3)$, $X_3 = \mathbb{Z}_9$, $\mathbb{B}_3$ 為 12個 3-subsets, 滿足…

(∘) A design is incomplete if for each $B \in \mathbb{B}$, $|B| < |X|$.

(Complete otherwise)

(∘) A design $(X, \mathbb{B})$ is balanced if $\forall\ x \in X$, the number of sets in $\mathbb{B}$ which contains x is a constant, i.e.

$r(x) = |\{B \mid x \in B, B \in \mathbb{B}\}| = r(y)$ for any two elements (varieties) (treatments) x and y in X. $r(x) = r(y) = r$: replication "number"

e.g. Both $(X_1, \mathbb{B}_1)$ and $(X_2, \mathbb{B}_2)$ are incomplete and balanced.

(∘) If all sets in $\mathbb{B}$ are of the same size, say k, then $(X, \mathbb{B})$ is called a "block design" (區組設計).

Again, both $(X_1, \mathbb{B}_1)$ and $(X_2, \mathbb{B}_2)$ are block designs.

(∘) We use $\lambda(x, y)$ to denote the number of sets in $\mathbb{B}$ which contain $\{x, y\}$. (frequency of pairs x and y)

If for any two distinct elements (t distinct) x and y, $\lambda(x, y)$ remains fixed, then we use λ to denote the frequency number of 2-design (t-design) $(X, \mathbb{B})$.

Per-Du

(*) In $(X_1, \mathbb{B}_1)$, $\lambda = 2$; In $(X_2, \mathbb{B}_2)$ and $(X_3, \mathbb{B}_3)$, $\lambda = 1$

(*) A 2-$_\lambda$design $(X, \mathbb{B})$ with $|X| = v$, $|\mathbb{B}| = b$, $|B| = k$ for each $B \in \mathbb{B}$, replication r and frequency λ is denoted as a 2-(v, b, r, k, λ)-design or 2-(v, k, λ) in short (?)

$\boxed{\text{Exercise 1-2 (3 points)}}$ Show that in a 2-(v, b, r, k, λ) design,

(1) $bk = vr$, (2) $\lambda(v-1) = r(k-1)$ and (3)* $b \geq v$. ← Important ideas!

1-4'

<u>Note</u> If the design defined above is a t-design, then we use $S_\lambda(t, k, v)$ to denote the design (or use t-(v, k, λ) in literatures).

<u>Fact 1</u>

In a 2-(v, k, λ) design, $b = \dfrac{\lambda v(v-1)}{k(k-1)}$ and $r = \dfrac{\lambda(v-1)}{k-1}$.

<u>Open problem</u>

From the graph theory point of view, if a 2-(v, k, λ) design exists, then $K_k | \lambda K_v$, i.e., λK_v can be decomposed into b edge-disjoint K_k's. This implies that $\frac{\lambda(v-1)}{k-1}$ and $\lambda\binom{v}{2}/\binom{k}{2}$ are integers. <u>Are these two conditions also sufficient?</u>

(※) 我们可以用 (0,1)-矩陣來表示一个设计 $(X, \mathbb{B})$，然後從矩陣的性質來研究"设计的性質"。

令 $X = \{x_1, x_2, \cdots, x_v\}$, $\mathbb{B} = \{B_1, B_2, \cdots, B_b\}$.

定義 $(X, \mathbb{B})$ 的關聯矩陣 (Incidence matrix) 如下：

$$N_{v\times b} = [n_{i,j}]_{v\times b}\ ,\quad n_{i,j} \in \{0,1\},\quad 1 \le i \le v,\quad 1 \le j \le b,$$

$n_{i,j} = 1$ if and only if $x_i \in B_j$.

$(X, \mathbb{B})$: $2-(v, k, \lambda)$ design.

$$N = \begin{bmatrix} & & \\ & n_{i,j} & \\ & & \end{bmatrix}_{v\times b}$$

單位矩陣 全"1"矩陣

$$N \cdot N^T = \begin{bmatrix} r & & \\ & r & \lambda's \\ & \lambda's & \ddots \\ & & r \end{bmatrix}_{v\times v} = (r-\lambda) I_v + \lambda J_v$$

$$\det(NN^T) = (r-\lambda+\lambda v)\cdot(r-\lambda)^{v-1} > 0$$

$\Uparrow$ $\Uparrow$ $v-1 > k-1$

$\vdots$ $r+\lambda(v-1)$

$= r + r(k-1) = rk$

$\operatorname{rank} N \ge v \Rightarrow b \ge v$

Per-Duc

(。) $(X_1, \mathbb{B}_1) \approx K_3 \mid 2K_4$.

$(X_2, \mathbb{B}_2) \approx K_3 \mid K_7$.

$(X_3, \mathbb{B}_3) \approx K_3 \mid K_9$.

N.C. for the existence of a 2-(v,k,λ) design:

(1) $k < v$, (Incomplete designs)

(2) $(k-1) \mid \lambda(v-1)$, and

(3) $\binom{k}{2} \mid \lambda \binom{v}{2}$.

(*) R. Wilson: The necessary conditions are also sufficient for large v.

(說明) 為了証明上述的必要條件也是充分條件，我們需要對所有滿足上述條件的 (v,k,λ) 都建構出一个 2-(v,k,λ) design. (這是比較困難的部份!)

$(X_1, \mathbb{B}_1)$ 是一个 2-$(4,3,2)$ design.

$(X_2, \mathbb{B}_2)$ 是一个 2-$(7,3,1)$ design.

$(X_2, \mathbb{B}_3)$ 是一个 2-$(9,3,1)$ design

Similarly we can determine the necessary conditions for the existence of a t-(v,k,λ) design.

(*) An $S_\lambda(t,k,v)$ exists. $\underset{N.C.}{\Rightarrow}$

(1) $v > k$,

(2) $\binom{k-1}{t-1} \mid \lambda\binom{v-1}{t-1}$, and $\left(r = \dfrac{\lambda\binom{v-1}{t-1}}{\binom{k-1}{t-1}} \right)$

(3) $\binom{k}{t} \mid \lambda\binom{v}{t}$。

Again, are these conditions also sufficient?

(**) 研究 $(t \geq 3)$-design 的存在問題，相較 $t=2$ 就要困難許多。

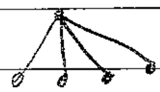
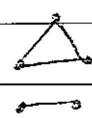
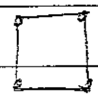
以下是兩個 $t=3$ 的例子.

1. $v = 2^n$, $n \geq 2$

$$X = (\mathbb{Z}_2)^n, \quad \mathcal{B} = \{\{\vec{x},\vec{y},\vec{z},\vec{w}\} \mid \vec{x}+\vec{y}+\vec{z}+\vec{w}=\vec{0},\ \vec{x},\vec{y},\vec{z},\vec{w} \in (\mathbb{Z}_2)^n\}.$$

2. $v = 10$

$$X = E(K_5), \quad \mathcal{B} = \{\{e_1,e_2,e_3,e_4\} \mid e_i \in E(K_5),\ i=1,2,3,4, \text{ and}$$

$\langle\{e_1,e_2,e_3,e_3\}\rangle \cong$  or  or  $\}$。

(*) 我们要学习的内容，主要是如何建構滿足給定條件的設計。