FGT Lecture 10

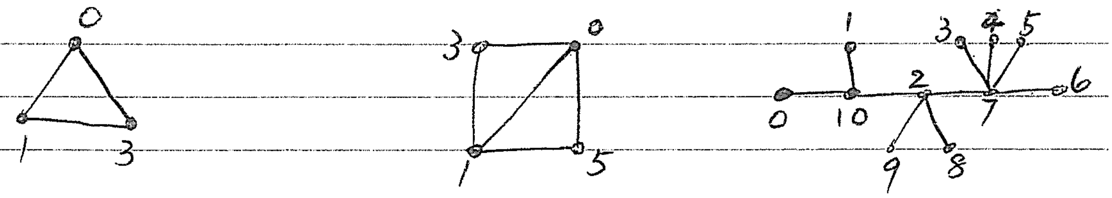
Graph Labelings

1. Graceful labeling

(o) A graceful labeling of a graph G is a 1−1 mapping f, $f: V(G) \xrightarrow{1-1} \{0, 1, 2, \cdots, \|G\|\}$ such that the weight of all of edges uv, defined by $|f(u) - f(v)|$, are distinct.

Examples

G :

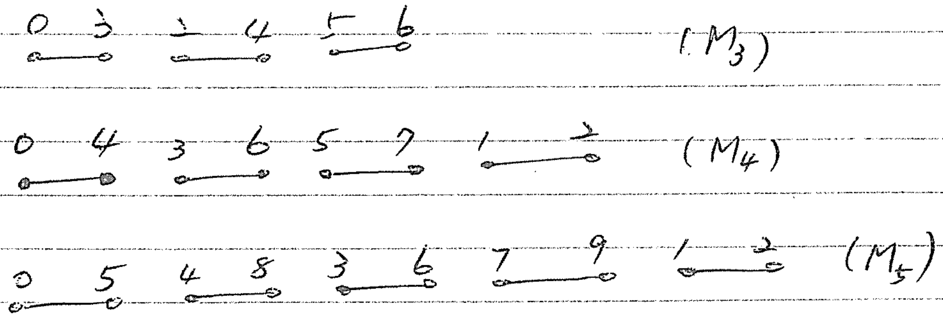


(oo) So, in general, we consider the graceful labeling of a connected graph since for tree $|G| = \|G\| + 1$. In order that f to be 1−1 $\|G\| \geq |G| - 1$.

(o) If $\|G\| < |G|$, then, we use $f: V(G) \xrightarrow{1-1} \{0, 1, 2, \cdots, |G|\}$ instead.

Example

G:



(°) A graph is graceful if it has a graceful (β) labeling.

β: Alex Rosa (1967)

Graceful: S. W. Golomb

<u>Conjecture</u> (Ringl-Kotzig) ~ 1980

Every tree is graceful.

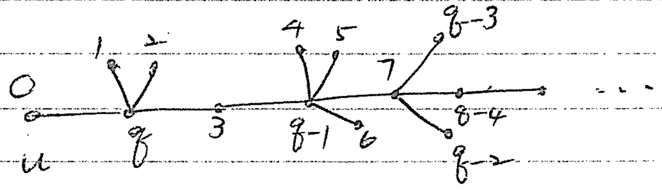
(*) Many results are obtained on special trees. But, no one can prove the conjecture so far.

<u>Theorem</u>

(1) Caterpillars are graceful.

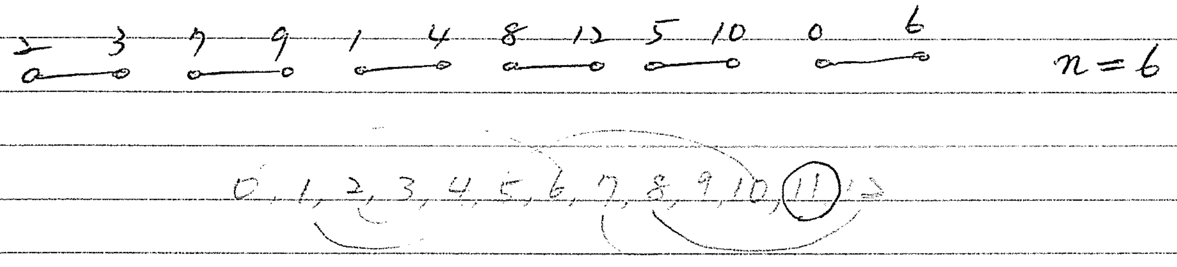
(2) M_n is graceful for all $n \in \mathbb{N}$.

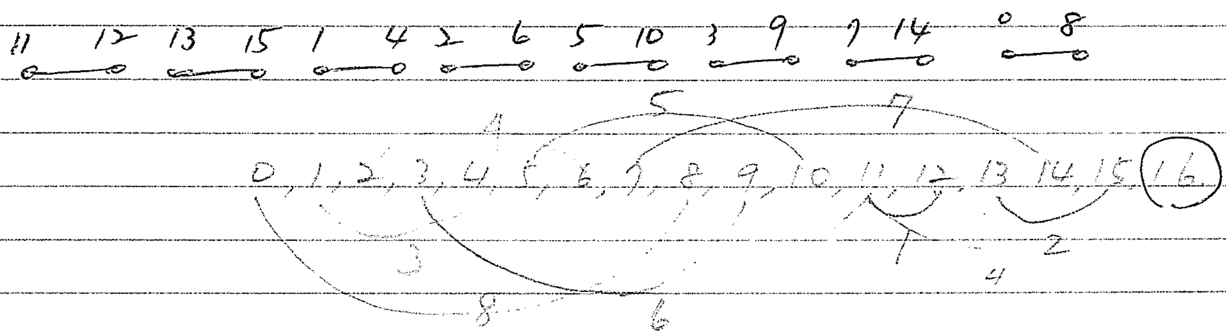
<u>Proof.</u> (1) The proof can be obtained by a direct labeling: starting from the end vertex u as shown in the following figure, let its label be 0, and then its neighbor be q, $\circ\circ\circ$. Then, all weights are distinct.



(2) The proof of this result is quite long.

The case when $n \equiv 0$ or $1 \pmod 4$ is known as Skolem sequence of type A and $n \equiv 2$ or $3 \pmod 4$ is Skolem sequence of type B. We shall give some examples here and omit the proof.





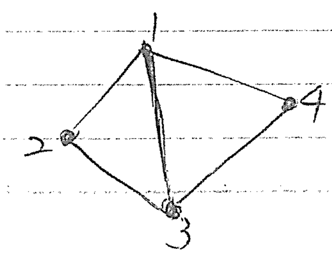
(o) In type A, 16 is not used and in type B, 11 is not used.

(*) Graceful labeling is very useful in constructing Combinatorial designs.

2. Prime labeling

(o) A prime labeling of a graph G of order n is a 1-1, onto mapping $f : V(G) \xrightarrow[\text{onto}]{1\text{-}1} \{1, 2, \cdots, n\}$ such that if $uv \in E(G)$, then $\gcd(f(u), f(v)) = 1$.

Example



Conjecture (Ringel).

Every tree has a prime labeling.

√(∘) Let G be a graph of order n. If $\alpha(G) < \lfloor \frac{n}{2} \rfloor$, then G has no prime labelings.

Proof. All even integers in $\{1, 2, \cdots, n\}$ form an independent set in a co-prime graph G_n.

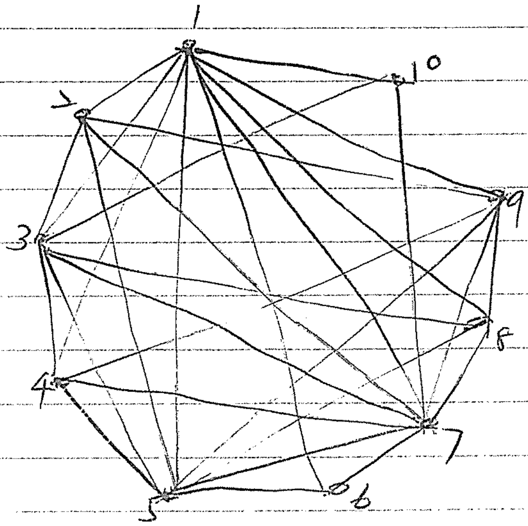
(*) Co-prime graph of order n, G_n.

$V(G_n) = \{1, 2, \cdots, n\} = [n]$.

$\{i, j\} \in E(G_n)$ if $i, j \in [n]$ and $\gcd(i, j) = 1$.

(∘∘) If a graph G of order n has a prime labeling, then $G \leq G_n$.

$n = 10$

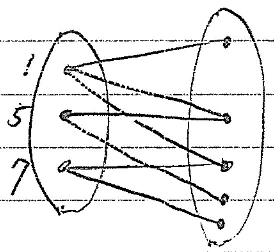


(*) For each n, there exists a prime in $(n, 2n)$. This is known as Bertrand's Postulate or sometimes Bertrand-Chebyshev Thm

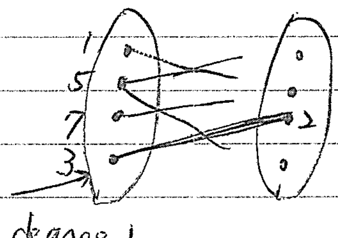
(*) The number of primes in $[n]$ is $\pi(n) \sim \frac{n}{\log n}$ (base e).

(◦) If T is a tree of order 8, then T has a prime labeling.

Proof. Case 1 $T = (A, B)$ and $|A| \leq 3$.



Case 2 $|A| = |B| = 4$



degree 1
(By Pigeon-hole principle)

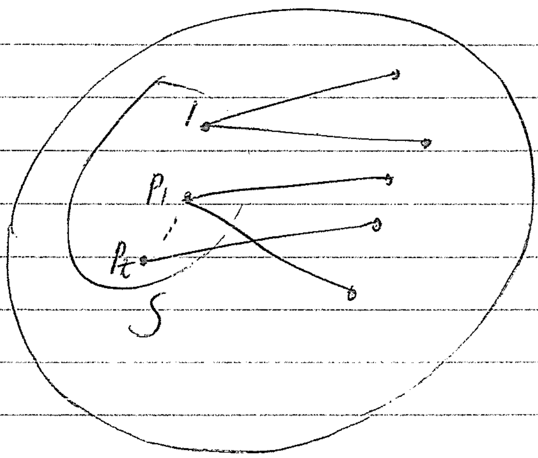
(◦) If G is a bipartite graph of order 8 and $G \ncong K_{4,4}$, then G has a prime labeling.

(◦) If G has a prime labeling and S is a set of edges in G, then $G - S$ has a prime labeling.

Let $P(a,b)$ denote the set of primes between integers a and b.

<u>Theorem</u> If G is a graph of order n and S is a vertex cover of G such that $|S| \leq P(\lceil \frac{n}{2} \rceil, n] + 1$, then G has a prime labeling.

<u>Proof.</u> Let S be labelled by 1, and primes in $P(\lceil \frac{n}{2} \rceil, n]$. Then, the proof follows. (Since S is a vertex cover, no edges exist in $\langle G-S \rangle_G$.) ▨



接 6', 6''

3. <u>Magic labelings</u> (on Edges)

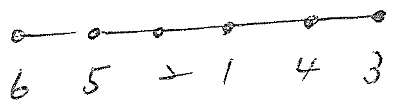
G has a <u>magic labeling</u> if there exists a 1-1 mapping $f: E(G) \longrightarrow \{1, 2, 3, \cdots, \|G\|\}$ such that for each v, the weight of v, defined by $\sum_{u \in N_G(v)} f(uv)$, is a constant.

(∘) If all weights are different, then we have an <u>anti-magic labeling</u>.

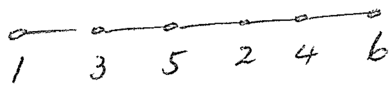
Prime sum labeling (Prime difference labeling)

A mapping $f : V(G) \xrightarrow[\text{onto}]{1\text{-}1} \{1, 2, \cdots, |G|\}$ is called a prime sum (difference) labeling of G if for each edge $uv \in E(G)$, $f(u)+f(v)$ ($|f(u)-f(v)|$) is a prime.

e.g.



prime sum labeling



prime difference labeling

Problem 1

Does C_n have a prime sum labeling for each $n \in \mathbb{N}$?

(1, 2, 3, 4), (1, 6, 5, 2, 3, 4), (1, 6, 7, 4, 3, 8, 5, 2), (1, 6, 7, 4, 3, 8, 5, 12, 11, 2)

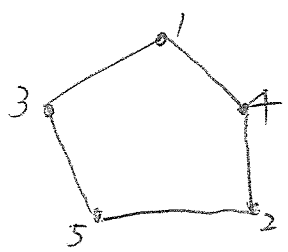
Problem 2

Does C_n have a prime difference labeling for each $n \in \mathbb{N}$?

Fact 1 If n is odd, then C_n does not have a prime sum labeling.

Fact 2 If $n = 4$, C_n has a prime sum labeling, but not a prime difference labeling.

Fact 3 If $n = 5$, then C_n has a prime difference labeling



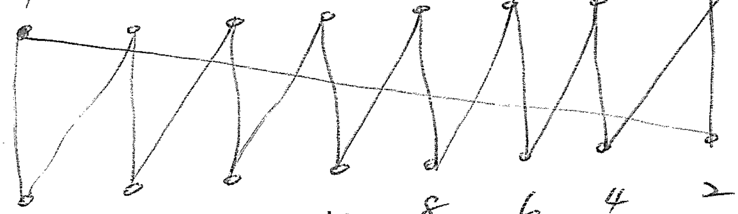
Proposition

If p and $p+2$ are primes, then C_{p-1} has a prime sum labeling provided $p \geq 5$.

Proof. The labeling can be obtained directly:

$$(1, 2, p-2, 4, p-4, 6, p-6, 8, \cdots, p-(p-5), p-3, 3, p-1).$$ ▨

e.g. $p = 17$



Conjecture For each even integer $n \geq 4$, C_n has a prime sum labeling.

Proposition

For each $n \geq 5$, the Hamilton path P_n has a prime difference labeling. cycle (C_n)?

Proof. First, we consider $n \in \{5, 6, 7, 8, 9\}$, the solutions are

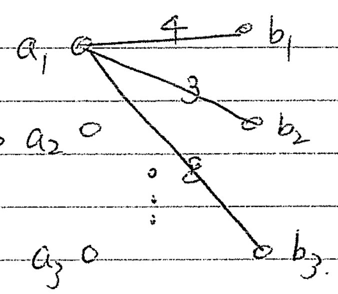
$\langle 1, 3, 5, 2, 4 \rangle$, $\langle 1, 3, 5, 2, 4, 6 \rangle$, $\langle 1, 3, 5, 2, 7, 4, 6 \rangle$, $\langle 1, 3, 8, 5, 2, 7, 4, 6 \rangle$ and $\langle 1, 3, 8, 5, 2, 7, 9, 4, 6 \rangle$. Then, the proof follows by put paths together. (?) ▨

<u>Example</u> For each $n=1$ and $n \geq 3$, $K_{n,n}$ has a magic labeling.

(∘) A magic labeling of $K_{n,n}$ is corresponding to the existence of a (weak) magic square (only rows and columns are required to be the same sum).

	b_1	b_2	b_3
a_1	4	3	8
a_2	9	5	1
a_3	2	7	6

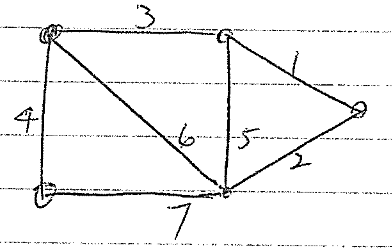
$\longleftrightarrow$



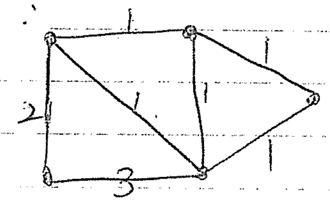
(∘) For $n \neq 2, 6$, we can use a pair of orthogonal Latin squares to construct a magic labeling of $K_{n,n}$.

7. Anti-magic labelings

G has an anti-magic labeling if there exists a 1-1 mapping $f: E(G) \to \{1, 2, \cdots, \|G\|\}$ such that for each v, the weights $\sum_{u \in N_G(v)} f(uv)$ are all different.

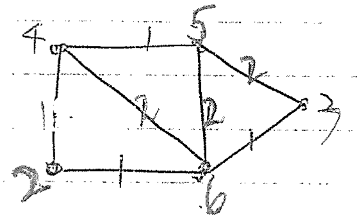


(◦) We may also use $f: E(G) \longrightarrow \{1, 2, \cdots, k\}$ to obtain an <u>anti-magic labeling.</u> <u>$k \leq \|G\|$ in general</u>



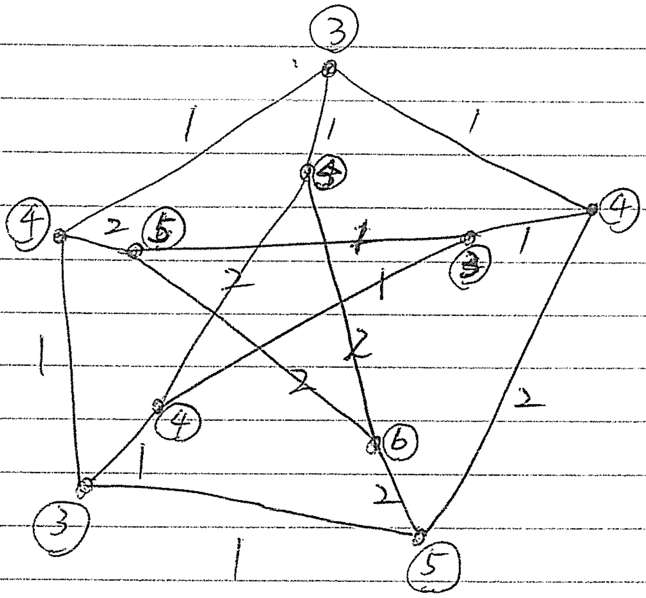
5. <u>Vertex-distinguished labeling</u>

We only require the weight of u is different from the weight of when uv is an edge of G.



<u>Conjecture</u> $k = 3$ is enough!

One more example :



Conjecture For all graphs G, a vertex-distinguished labeling can be obtained by a $\{1,2,3\}$-labeling. (Use 1, 2, 3 only!)

基礎圖论介绍到這裏,希望你(好)有興趣,继续学习。

Happy New Year!

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2019, 1, 2.