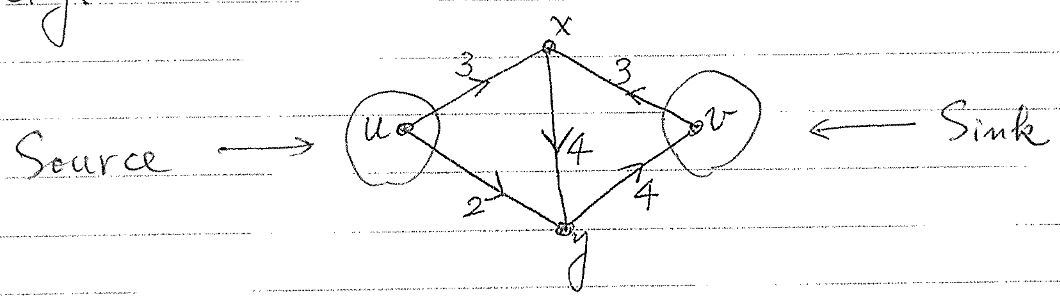
FGT Lecture 6 Nov. 5-

Networks

Definition

A network N is a digraph D with two distinguished vertices u and v, called the source and sink of N, resp., and a nonnegative integer-valued function c on $A(D)$, i.e. $c : A(D) \to \mathbb{R}^+ \cup \{0\}$. The digraph is called the underlying digraph of N. The function c is the capacity function of N and its value $c(\vec{a}) = c(x,y)$ on the arc $\vec{a} = (x,y)$ is referred to as the capacity of $\vec{a}$.

e.g.



Definition (Flow)

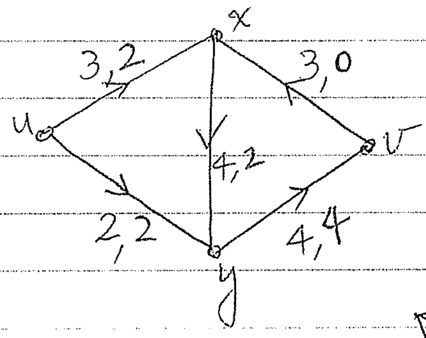
A flow in a network N (with underlying digraph D, source u, sink v, and capacity function c) is an integer-valued function f on $A(D)$ such that for every $\vec{a} \in A(D)$,

$0 \le f(\vec{a}) \le c(\vec{a})$, and

$$\sum_{y \in N^+(x)} f(x,y) = \sum_{y \in N^-(x)} f(y,x) \quad \forall \quad x \in V(D) \setminus \{u,v\}. \quad \cdots\cdots (*)$$

(*) is known as the "conservation equation" of the network.

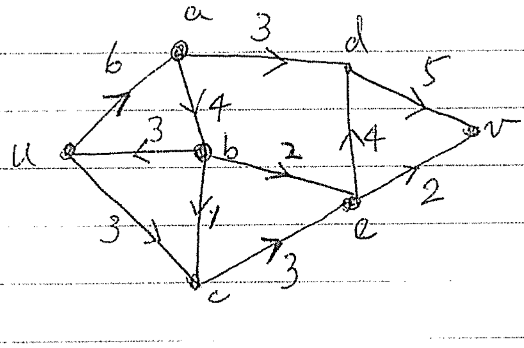
Example of network with flow (1)



val f : The value of a flow f in a network, which is defined as the net flow out of the source of N; equivalently, the net flow into the sink of N.

$\boxed{\text{val } f = 4}$

Another example (2)



What is "maximum flow" from u to v ?

Notation (X, Y)

$$(X,Y) = \{(x,y) \in A(D) \mid x \in X,\ y \in Y\}.$$

Definition (Cut)

Let N be a network with underlying digraph D, source u, sink v. A cut in N is $(X, \overline{X})$ where $\overline{X} = V(D) \setminus X$, $u \in X$ and $v \in \overline{X}$.

Definition (Capacity of a cut)

Let $K = (X, \overline{X})$ be a cut of network N with capacity c. Then, the capacity of the cut K, denoted $\operatorname{cap} K = c(X, \overline{X}) = \sum_{(x,y) \in (X, \overline{X})} c(x,y)$.

In example (2).

Let $X = \{u, a, b, c\}$, then $c(X, \overline{X}) = 6$.

$\overline{X} = \{d, e, v\}$.

Theorem Let f be a flow in a network N and let $K = (X, \overline{X})$ be a cut in N. Then

$$\operatorname{val} f = f(X, \overline{X}) - f(\overline{X}, X) \le \operatorname{cap} K.$$

Minimum cut

A cut K in a network N is called a minimum cut if $\operatorname{cap} K \leq \operatorname{cap} K'$ for every cut K' in N.

(Fact) Let f be a flow and K be a cut in N. If $\operatorname{val} f = \operatorname{cap} K$, then f is a maximum flow and K is a minimum cut.

(A flow f in N is called a maximum flow if $\operatorname{val} f \geq \operatorname{val} f'$ for every flow f' in N.

Proof. It can be proved (?) that if f^* is a maximum flow and K^* is a minimum cut, then $\operatorname{val} f^* \leq \operatorname{cap} K^*$. Now, f is a flow, hence $\operatorname{val} f \leq \operatorname{val} f^*$; K^* is a minimum cut, hence $\operatorname{cap} K^* \leq \operatorname{cap} K$. Since $\operatorname{val} f = \operatorname{cap} K$ and

$$\operatorname{val} f \leq \operatorname{val} f^* \leq \operatorname{cap} K^* \leq \operatorname{cap} K,$$ we conclude that $\operatorname{val} f = \operatorname{val} f^*$ and $\operatorname{cap} K = \operatorname{cap} K^*$. ∎

补充资料

(•) If there exists a K and an f s.t. $cap\,K = val\,f$, then K is a minimum cut and f is a maximum flow.

Theorem. (Ford and Fulkson, 1956-1962)

In any network N defined on D, the value of a maximum flow equals the capacity of a minimum cut.

Proof.

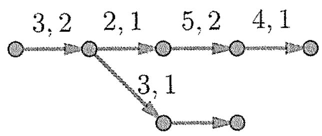
Clearly, if there exist no cuts such that its capacity of the cut is $val\,f$, then f does not exist. (Theorem 21) So, it suffices to claim that if the value of a maximum flow f is v, then there exists a cut K, such that $cap\,K = val\,f = v$.

Define a subset $S \subseteq V(D)$ recursively as follows. Let $s \in S$. If $x \in S$, and $c(x,y) > f(x,y)$ or $f(y,x) > 0$, then let $y \in S$. We shall prove that (S, S') is a cut with capacity v. First, we claim $t \notin S$. Suppose not, i.e., $t \in S$. Hence, we can find a sequence of vertices in N such that $s = x_0, x_1, \cdots, x_l = t$. Moreover, if we let $\varepsilon_i = max\{c(x_i, x_{i+1}) - f(x_i, x_{i+1}), f(x_{i+1}, x_i)\}$, $i = 0, 1, \cdots, l-1$, then $\varepsilon_i > 0$. Let $\varepsilon = min\{\varepsilon_i\}$. Now, let $f^*(x_i, x_{i+1}) = f(x_i, x_{i+1}) + \varepsilon$ if $c(x_i, x_{i+1}) - f(x_i, x_{i+1}) = \varepsilon_i > 0$ and $f^*(x_{i+1}, x_i) = f(x_{i+1}, x_i) - \varepsilon$ if $f(x_{i+1}, x_i) = \varepsilon_i > 0$. As a consequence, f^* is a flow from s into t with value $val\,f^* = v + \varepsilon$, a contradiction. (See Figure 1) By the definition of a flow,

$val\,f = v = \sum_{x \in S, y \in S'} f(x,y) - \sum_{x \in S', y \in S} f(x,y)$. ——(1)

6-4''

Again, by the definition of S, if $x \in S$ and $y \in S'$, then $c(x,y) = f(x,y)$ and $f(y,x) = 0$. This implies that (1) = $\sum_{x \in S, y \in S'} c(x,y) = cap\,(S,S') = v$, the proof follows. ■



$\varepsilon = 1$

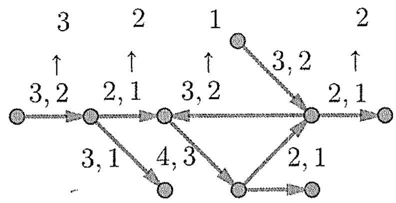
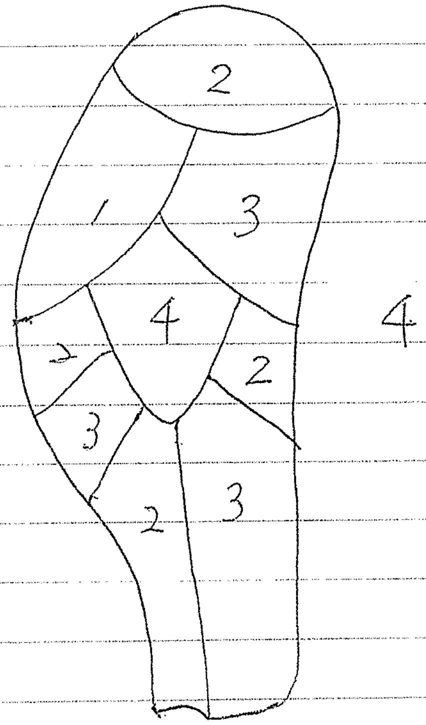


Figure : augmenting flow f^*

NO. 6-5
DATE

Graph Coloring



Map Coloring Problem

1852, Oct. 23, Augustus De Morgan $\xrightarrow{\text{sent a letter}}$ William Rowan Hamilton
(University College, London) (Trinity College, Dublin)

... My pupil says he guessed $\boxed{\text{4 colors}}$ are enough to color the compartments differently so that figures with any portion of common boundary are differently colored. ...
(of England)

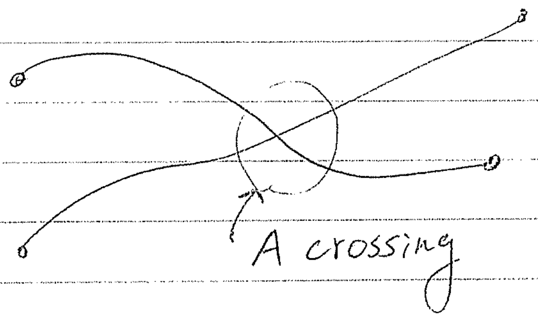
Student : Frederick Guthrie

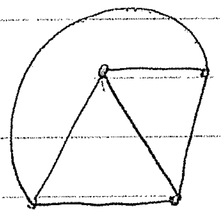
Learned this idea from his brother : Francis Guthrie
(He provided an incomplete proof.)

Definition (Region Colorable)

A plane graph G is said to be n-region colorable if the regions of G can be colored with n or fewer colors so that adjacent regions are colored differently.

Note A plane graph G is a graph which can be drawn on a plane such that no two edges have a crossing.





K_4 is a plane graph, but not K_5.

Remark We shall talk about crossings of graphs later.

The Four Color Conjecture

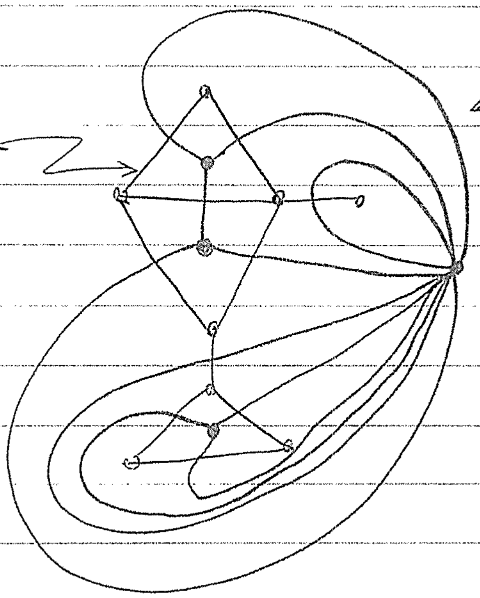
Every map (plane graph) is 4-region colorable.

Now, it is known as the 4-color Theorem, 4CT in short.

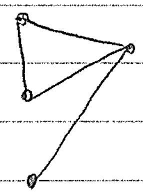
Pseudograph G^*

(With loops and multi-edges)

G



$\tilde{G}^*$:



Step 1 For each region of a plane graph, define a vertex.

Step 2 Join two (one) vertices with an edge (or loop) if they have a common boundary.

Step 3 This pseudograph (G^*) is known as the dual of the plane graph (G).

Fact $(G^*)^* \cong G$.

Fact $\tilde{G}^*$ is called the underlying graph of G^* if loops are deleted and replace the multiple edges by a single edge.

So, we can use the vertex coloring of $\tilde{G}^*$ to color the regions of a plane graph G.

Definition (Vertex coloring)

A vertex k-coloring φ is a mapping $\varphi: V(G) \to [1,k]$ such
(k-coloring in short)
that $\varphi(u) \neq \varphi(v)$ if uv is an edge of G. The minimum number k such that G has a k-coloring is called the chromatic number of G denoted by $\chi(G)$. (If G has a k-coloring, G is said to k-colorable.)

(Fact 1) If $G \geq K_k$, then $\chi(G) \geq k$.

(Fact 2) If G is a bipartite graph, then $\chi(G) = 2$ provided G contains at least one edge.

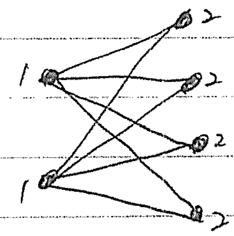
(Fact 3) If G has a k-coloring $\varphi: V(G) \to \{1, 2, \ldots, k\}$, then for each $c \in [1,k]$, $\varphi^{-1}(c) \subseteq V(G)$ is an independent set.

(Fact 4) $\chi(K_n) = n$ and $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even; and} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

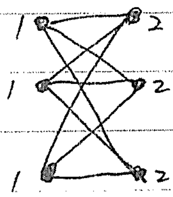
(Fact 5) If $h \geq k$ and G is k-colorable, then G is also h-colorable.

Remark 1 If G has a k-coloring φ such that $\left||\varphi^{-1}(c)| - |\varphi^{-1}(d)|\right| \leq 1$ for any $c, d \in [1, k]$, then φ is an equitable k-coloring.

Remark 2 If G has a k-coloring, then G "may not" have an equitable k-coloring or equitable $(k+1)$-coloring.



$\chi(K_{2,4}) = 2$



$\chi(K_{3,3}) = 2$

No equitable 3-coloring of $K_{3,3}$ exists. (?)

Definition (critically n-chromatic)

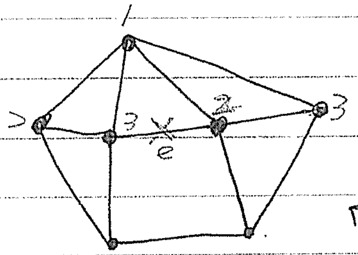
For each $n \geq 2$, G is critically n-chromatic if $\chi(G) = n$ and $\chi(G - v) = n - 1$ for all $v \in V(G)$.

Example K_n is critically n-chromatic.

C_{2k+1} is critically 3-chromatic.

Definition (Minimally n-chromatic)

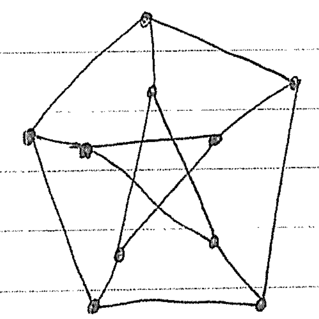
A graph G is minimally n-chromatic if $\chi(G)=n$ and $\chi(G-e)=n-1$ for all edges $e \in E(G)$.



Not minimally 4-chromatic

$\chi(G-e)=4$

Critically 4-chromatic



Not minimally 3-chromatic

Not critically 3-chromatic

Theorem (Brooks) 1941

If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \le \Delta(G)$.

Proof. (Omit here.)   (*) It is easier to consider the connected graphs which are not regular.

Theorem (Halin, 1967 Szekeres and Wilf)

For every graph G, $\chi(G) \le 1 + \max \delta(G')$, where the maximum is taken over all induced subgraph G' of G. (Notice that if G is not a regular graph, then $\chi(G) \le 1 + \delta(G) \le \Delta(G)$.)

Proof. First, we claim that if G is critically n-chromatic, then $\delta(G) \ge n-1$. Suppose not. Let $\delta(G) \le n-2$ and $\deg(v) = \delta(G)$ where $v \in V(G)$. By assumption, $\chi(G-v) = n-1$. Now, G has an $(n-1)$-coloring since v can be colored with the color missing in its neighbors at most $(n-2)$ vertices. This is a contradition to $\chi(G) = n$.

Now, we are ready for this theorem. Let H be an induced subgraph of G which is n-critical. (We can find this graph by deleting vertices if necessary. So, $\delta(H) \le \max_{G' \le G} \delta(G')$. By the fact that $\delta(H) \ge n-1$, $\max_{G' \le G} \delta(G') \ge n-1 = \chi(G) - 1$.

Hence $\chi(G) \le 1 + \max_{G' \le G} \delta(G')$.

<u>Theorem</u> (Gallai, 1968)

For any graph G, $\chi(G) \le 1 + m(G)$, where $m(G)$ denotes the length of a longest path in G. (If G does have a Hamilton path, then $\chi(G)$ is smaller.)

<u>Proof.</u> If H is an n-critical induced subgraph of G, then $\delta(H) \ge n-1$ and therefore H contains a path of length $n-1$. This implies that $m(G) \ge n-1 = \chi(G) - 1$ and $\chi(G) \le m(G) + 1$.

Greedy Coloring Algorithm

Let $\{1, 2, \cdots, k\}$ be the colors we plan to use. Color a vertex with smaller integer if it is available.