FGT Lecture 5 Oct. 18 —

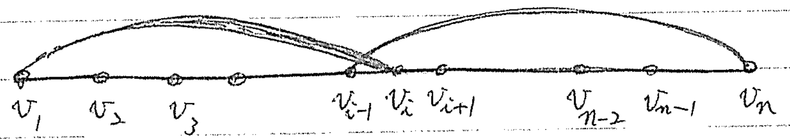
Theorem (Ore, 1960)

If G is a graph of order $n \geq 3$ such that for all distinct non-adjacent vertices u and v, $\deg(u) + \deg(v) \geq n$, then G contains a Hamilton cycle.

Proof. (By maximality argument, 最大反例的概念)

Assume the assertion is false. Then, there exists a nonhamiltonian graph $\tilde{G}$ of order $n \geq 3$ which satisfies the hypothesis of the theorem. Therefore, for any two distinct vertices v_1 and v_2, $\tilde{G} + v_1v_2$ contains a Hamilton cycle. Furthermore, every Hamilton cycle of $\tilde{G} + v_1v_2$ contains the edge v_1v_2.

Now, let u and v be two nonadjacent vertices of G. Since $G + uv$ contains a Hamilton cycle, G contains a Hamilton path $\langle u = v_1, v_2, \cdots, v_n = v \rangle$.



By observation, if $v_1v_i \in E(G)$, $2 \leq i \leq n$, then $v_{i-1}v_n \notin E(G)$. (?)

For otherwise, we have a Hamilton cycle: $(v_1, v_i, v_{i+1}, \cdots, v_n, v_{i-1}, v_{i-2}, \cdots, v_1)$.

This implies that if $\deg(v_1) = t$, $\deg(v_n) \leq (n-1) - t$. Hence,

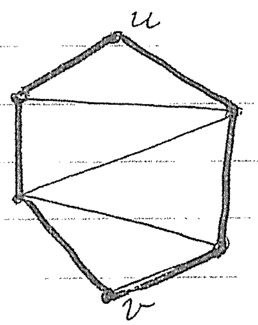
$\deg(u) + \deg(v) \leq t + (n-1) - t = n-1$. $\rightarrow\leftarrow$ We conclude

that G contains a Hamilton cycle.

Remark

There are sufficient conditions (Quite a few!) for the existence of Hamilton cycles in a graph, but so far, none of them is also necessary.

For example, the condition in the above theorem is not necessary.



$\deg(u) + \deg(v) = 4 < 6$.

(***) Finding a cycle with prescribed length (k) is in general very difficult. (Algorithmic thinking)

Note A graph with Hamilton cycles is called a hamiltonian graph.

Weighted graphs

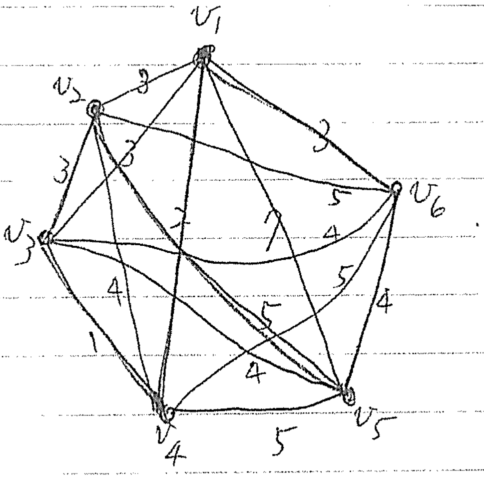
Definition (Weighted graphs)

A graph G is weighted if each edge is assigned a weight by using a weighted function $w: E(G) \longrightarrow \mathbb{R}$.

Traveling Salesman Problem (TSP)

In a weighted complete graph G, find a minimum Hamilton cycle, i.e, the sum of all weights in the cycle is minimum comparing the sums of all the weights of the other Hamilton cycles.

e.g.



~~How about~~ minimum ~~spanning~~ trees? See 5-3'.

(Note) If each weight is a finite number, then a $\boxed{\text{greedy}}$ algorithm can provide an answer. (May not be minimum.)

Connectivity

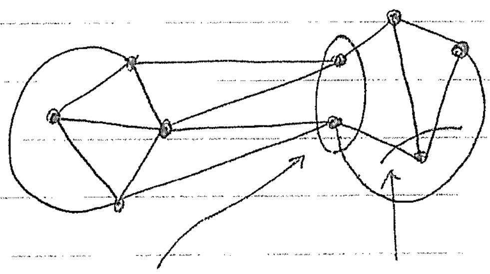
Definition (Connectivity)

The connectivity of a graph G, $\kappa(G)$, is the minimum number of vertices whose removal from G results in a disconnected graph or a trivial graph (a graph with one vertex).

Definition (Edge connectivity)

The edge connectivity of a graph G, $\kappa_1(G)$, is the minimum number of edges whose removal from G results in a disconnected graph.

G :



$\kappa(G) = 2$, $\kappa_1(G) = 3$.

Theorem For any graph G, $\kappa(G) \le \kappa_1(G) \le \delta(G)$.

Proof. Let $v \in V(G)$ and $\deg(v) = \delta(G)$. Then, the deletion of all edges incident to v results in a disconnected graph.

Hence, $\kappa_1(G) \leq \delta(G)$. Now, consider the other inequality. First, if $\kappa_1(G) = 0$, then the G is already disconnected, hence $\kappa(G) = 0$. Assume that $\kappa_1(G) > 0$ and let E' be a set of $\kappa_1(G)$ edges such that $G - E'$ is disconnected. Let S be a set of vertices choosen from the set of vertices incident to edges in E' such that edge in incident to S exactly once. Therefore $|S| \leq |E'|$. Also, $G - S$ is disconnected or a trivial graph since $G - E'$ is disconnected. This implies that $\kappa(G) \leq |S| \leq |E'| = \kappa_1(G)$. ∎

<u>Problem</u> Let $a \leq b \leq c$ be positive integers. Then, there exists a graph G such that $\kappa(G) = a$, $\kappa_1(G) = b$ and $\delta(G) = c$.

<u>Definition</u> (n-connected and n-edge-connected)

A graph G is said to be n-connected (resp. n-edge-connected) if $\kappa(G) \geq n$ (resp. $\kappa_1(G) \geq n$).

(Fact) A graph is n-edge-connected if it is n-connected.

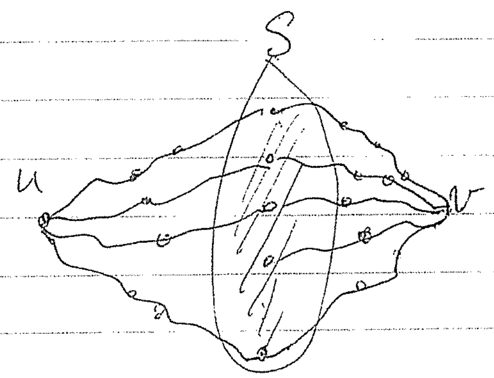
Menger's Theorem

Definition (Separating set)

A set of vertices S of a graph G is said to be a separating set of two vertices u and v of G if $G-S$ [(u,v)-separating set] is a disconnected graph in which u and v lie in different components. We also say S separates u and v.

Theorem (Menger, 1927)

Let u and v be nonadjacent vertices in G. Then, the minimum number of vertices that separates u and v is equal to the maximum number of internally disjoint u-v paths in G.



Proof. Many different versions.

Remark

Consider the existence of a Hamilton cycle, it is easy to see that if a graph is of larger connectivity, Then the probability of finding a Hamilton cycle is larger. The following theorem is one of this kind.

Theorem (Chvátal and Erdös, 1972) (Not prove in the class.)

Let G be a graph with at least three vertices. If $\kappa(G) \geq \beta(G)$, where $\beta(G)$ is the independence number G, then G contains a Hamilton cycle.

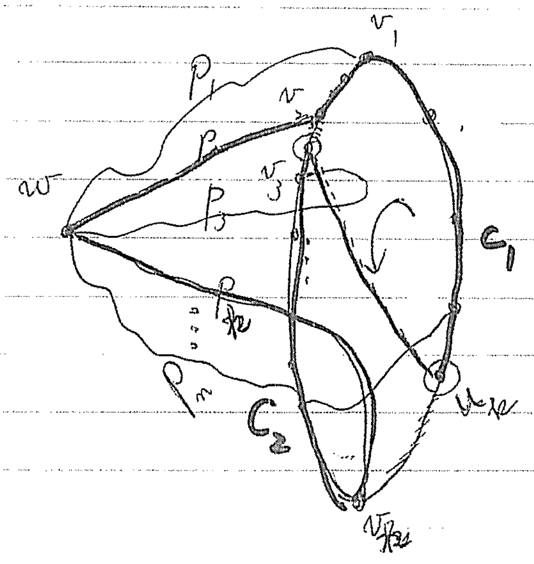
cycle exists

This implies that $\kappa(G) \geq 2$

Proof. Clearly, we may assume that $\beta(G) \geq 2$. Let C be the largest (size) cycle we can find in G. If $m = |C| = |G|$, then we are done. Otherwise, there is a vertex $w \notin V(C)$. We claim there exists a cycle of length at least $m+1$.

(n-connected $\Rightarrow |C| \geq n$. Let $V(C) \supseteq \{v_1, v_2, \cdots, v_n\}$.)

Let $\kappa(G) = n$. Then, by Menger's Theorem, we can find n internal-disjoint paths from w to v_i's, say $P_1, P_2, \cdots, P_n$. $(n \leq m)$ $(v_i \in V(C), i = 1, 2, \cdots, n)$



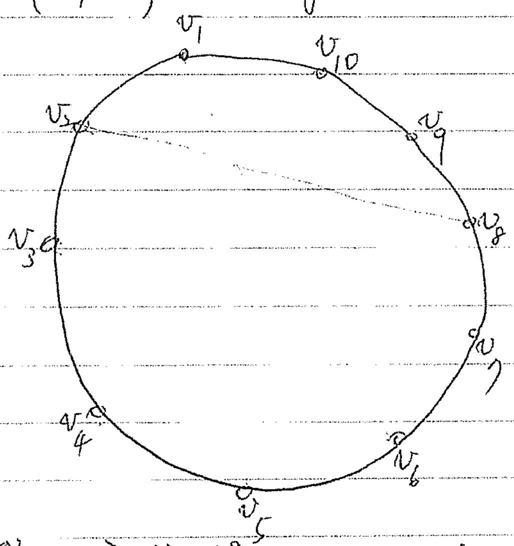
Now, there must be some vertices between v_i and v_{i+1} for $i = 1, 2, \cdots, n-1$. For otherwise, we already have a cycle of larger size. Let $u_i \sim v_i$ in counterclockwise direction, $i = 1, 2, \cdots, n$. Then, $\underline{u_i}$ is not incident to $\underline{w}$ by a similar reason as above. Let $S = \{w, u_1, u_2, \cdots, u_n\}$. Then S must be an independent set. Suppose not, let $u_j u_k$ be an edge in G. Then we have a larger cycle: $w \xrightarrow{P_j} v_j - C_1 - u_k u_j - C_2 - v_k \xrightarrow{P_k} w$. This concludes the proof.

(*) Finding the independence number of a graph is very difficult in general.

<u>Problem</u> Does Petersen graph contain a Hamilton cycle?

Answer: No.

<u>Proof.</u> Notice that Petersen graph P is 3-regular with girth 5, i.e. P contains no C_3's or C_4's. Now, if P contains a Hamilton cycle C, then $P \setminus C$ is a matching (perfect) with 5 edges. Let $C = (v_1, v_2, \cdots, v_{10})$. Clearly, the edge incident to v_1 must be incident to one of $\{v_5, v_6, v_7\}$. So, we consider the cases $v_1 \sim v_5$ and $v_1 \sim v_6$. ($v_1 \sim v_7$ is a symmetric case of $v_1 \sim v_5$.)



$$v_1 \sim v_5 \Rightarrow \begin{cases} v_2 \sim v_7 \Rightarrow v_3 \sim ? \text{ (No where to go!)} \\ v_2 \sim v_8 \Rightarrow v_3 \sim ? \text{ (No where to go!)} \end{cases}$$

$$v_1 \sim v_6 \Rightarrow v_2 \sim v_8 \Rightarrow v_3 \sim ? \text{ (No where to go!)}$$

▨

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Bonus Problem (2 points)

Let G be an n-connected graph where $n \geq 2$. Prove or disprove that any n distinct vertices can be included in a cycle of G.

Hint. By induction on n if the assertion is true.

Remark: If we are dealing with digraph, then a Hamilton cycle is a directed Hamilton cycle.

Tournament

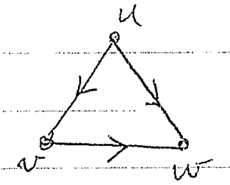
Definition (Symmetric digraph)
(asymmetric or oriented)

A digraph D is called symmetric if, whenever (u,v) is an
(asymmetric or oriented)
arc of D, then (v,u) is also an arc of D.
(not an arc of D)

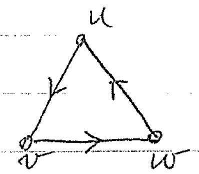
Note that an oriented graph D can be obtained from a graph G by assigning a direction to each edge of G, and thereby transforming G into an asymmetric digraph.

<u>Definition</u> (Tournament)

A complete asymmetric digraph is called a tournament. A tournament with p vertices will be denoted by T_p. A tournament T_p is <u>transitive</u> if for any u, v, w, (u, w) is an arc of of T_p provided (u, v) and (v, w) are arcs of T_p.



(Transitive)



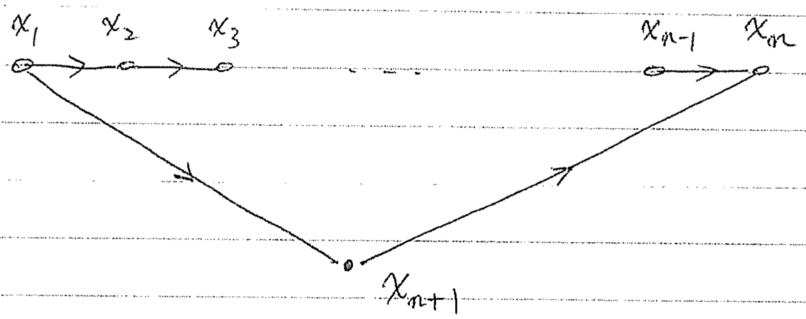
(not transitive)

<u>Theorem</u>

For any tournament T_p, T_p contains a directed Hamilton path.

<u>Proof.</u> (Maximality argument)

Let $\langle x_1, x_2, \cdots, x_n \rangle$ be the longest directed path which we can find in T_p. If $n = p$, then we have the proof. Otherwise, let $n < p$ and let x_{n+1} is not on the path. (See Figure below)



Then, (x_1, x_{n+1}) and (x_{n+1}, x_n) are arcs in T_p. If any of these two arcs are in different orientation, we have a longer path. Now, consider the orientations of the arcs between $\{x_2, x_3, \cdots, x_{n-1}\}$ and x_{n+1}. (x_2, x_{n+1}) must be an arc in T_p, otherwise, $\langle x_1, x_{n+1}, x_2, \cdots \rangle$ will be longer. Similarly, (x_3, x_{n+1}) is an arc in T_p, so are $(x_4, x_{n+1}), \cdots, (x_{n-1}, x_{n+1})$. But, now we have a longer path $\langle x_1, x_2, \cdots, x_{n-1}, x_{n+1}, x_n \rangle$, a contradiction. This concludes that $n = p$ and the proof. ∎

<u>Question</u> When does a tournament contain a directed Hamilton cycle?

Theorem

If T_p is a transitive tournament, then all vertices have distinct out-degrees.

Proof. It suffices to claim that no two vertices have the same out-degree. Suppose not. Let $deg^+(u) = deg^+(v)$ (in T_p), and W.L.O.G. $(u,v) \in T_p$. First, let $W = N^+(v)$ (out-neighbors). Then, $|W| = deg^+(v)$. By transitivity of T_p, for each $w \in W$, (u,w) is also an arcs of T_p. ($(u,v) \in T_p$ and $(v,w) \in T_p$)) This implies that $deg^+(u) \geq |W|+1 = deg^+(v)+1$. $\rightarrow\leftarrow$ ∎

Note

1. This kind of tournaments is good, since we shall have a unique "king" (maximum out-degree).
2. A transitive tournament does not contain any directed cycles! (Easy to prove.)