The existence of paths and cycles, $\boxed{\text{Distance in Graphs}}$

Proposition 1

Every graph contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$ (provided $\delta(G) \geq 2$).

Proof. Let $\langle x_1, x_2, \cdots, x_k \rangle$ be the longest path we can find in G. Then $N_G(x_k) \subseteq \{x_1, x_2, \cdots, x_{k-1}\}$. This implies that $\delta(G) \leq \deg_G(x_k) \leq k-1$, so, $k \geq \delta(G)+1$. This concludes the first part.

Now, if $\delta(G) \geq 2$, hence $\deg_G(x_k) \geq 2$. Let i be the smallest index in $\{1, 2, \cdots, k-1\}$ such that $x_i x_k \in E(G)$. This implies that $C = (x_i, x_{i+1}, \cdots, x_k)$ is a cycle in G. Since $\deg_G(x_k) \geq \delta(G)$, $i \leq k-\delta$, so, $k-i \geq \delta$. Since the cycle C has $k-i+1$ vertices, C has at least $\delta+1$ vertices. ∎

(Algorithmic aspect)

Remark Determining the length of the longest shortest path(s) in G is also known as "finding the diameter" which is not a difficult problem. But, finding the longest cycle in a graph is not easy at all.

How to determine the diameter of a graph G?

策略一

For each $v \in V(G)$, find $ecc(v)$. Then, $diam(G)$ is equal to $\max\{ecc(v) \mid v \in V(G)\}$.

策略二 (同学提供)

Pick an arbitrary vertex $u \in V(G)$ and find a vertex u' such that $d(u,u') = ecc(u)$. Then, $diam(G)$ is equal to $ecc(u')$.

Note that if this idea is good, then we have a quicker way to find $diam(G)$.

Proof. Give it a try!

策略三, Use adjacency matrix of G.

Problem Find the diameter of a graph G.

(Using the adjacency matrix of G, A.)

Fact 1 Let $V(G) = \{ v_i \mid i = 1, 2, \cdots, n \}$.

If $d(v_i, v_j) = h$, then $A^h(i,j) \neq 0$. $(h \geq 1)$

Fact 2

If $d(v_i, v_j) = h$, then $A^{h'}(i,j) = 0$ for each $1 \leq h' < h$.

Fact 3

Let $\mathrm{diam}(G) = t$ be the diameter of G. Then, for each $t' < t$, there exist i and j such that $A^{t'}(i,j) = 0$.

Fact 4

For all $i \neq j$, there exists an $1 \leq s \leq t$ such that $A^s(i,j) \neq 0$. (Clearly, let $s = d(v_i, v_j)$.)

* **(Fact 5)**

Let $\mathcal{A} = \sum_{k=1}^{t} A^k$. Then, $\mathcal{A}(i,j) > 0$. Moreover, if s is the smallest positive integer such that $\left(\sum_{k=1}^{s} A^k\right)(i,j) > 0$, then $s = \mathrm{diam}(G)$.

Distance in Graphs

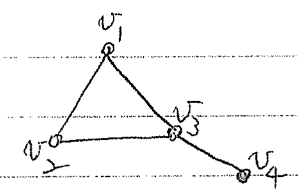
Definition (Adjacency Matrix)

Let G be a graph with $V(G)=\{v_1, v_2, \cdots, v_n\}$. Then the adjacency matrix A_G of the graph G is an $n\times n$ matrix such that

$$A_G(i,j)=\begin{cases}1 & \text{if } v_i \sim_G v_j \text{; and}\\ 0 & \text{otherwise.}\end{cases}$$

Example.

G :



$$A_G : \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 \end{array}$$

Remark

1. A_G is a symmetric matrix with "0"-diagonal.
2. The eigenvalues of A_G can be "found" and they are reals.

(*) The study of the relationship of "eigenvalues of G" (for distinct matrices of G) and "structure of G" is known as "Algebraic Graph Theory".

3. We are interested in "distance", so, we focus on A_G^k.

$$\begin{bmatrix}0&1&1&0\\1&0&1&0\\1&1&0&1\\0&0&1&0\end{bmatrix}\begin{bmatrix}0&1&1&0\\1&0&1&0\\1&1&0&1\\0&0&1&0\end{bmatrix}=\begin{bmatrix}2&1&1&1\\ \boxed{1\;\;2\;\;1\;\;1}\\1&1&3&0\\1&1&0&1\end{bmatrix}\begin{bmatrix}0&1&1&0\\1&0&1&0\\1&1&0&1\\0&0&1&0\end{bmatrix}=\begin{bmatrix}2&3&4&1\\3&2&4&1\\4&4&2&3\\1&1&3&0\end{bmatrix}$$

$v_2 \longrightarrow v_3$
3-walk

除了上面所介绍的相鄰矩陣 (Adjacency matrix) 之外尚有其它的表示法，以下介绍兩种比較常用的矩陣。

1. Incidence matrix (鄰接矩陣)

令 $G=(V,E)$, $V=\{v_1, v_2, \cdots, v_n\}$, $E=\{e_1, e_2, \cdots, e_m\}$。則 G 的 Incidence matrix $B(G)=[b_{ij}]_{n\times m}$，為(0,1)-矩陣，其中 $b_{ij}=1$ 若且唯若 $v_i \in e_j$, $i=1,2,\cdots,n$; $j=1,2,\cdots,m$。

(◦) 如果 G 為 simple graph，則每一行 (column) 的和皆為 2。

(◦) 如果 G 為 hypergraph (超圖) 則當 e_j 有 t 個点時，第 j 行和為 j，亦即，有 j 個 "1"。

例如： $V=\{v_1, v_2, v_3, v_4, v_5\}$, $e_1=\{v_1, v_3, v_4\}$, $e_2=\{v_3, v_5\}$, $e_3=\{v_3, v_4, v_5\}$, $e_4=\{v_1, v_2, v_4, v_5\}$，則

$$B(G)=\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(◦) $B(G)$ 除了在圖論有很多應用之外，在組合設計 (Combinatorial Design) 也扮演非常重要的角色，尤其是超圖的形式。

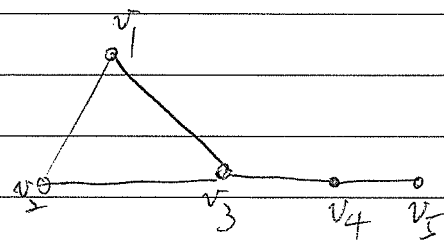
2. Laplacian matrix

如果 $G=(V,E)$, $V=\{v_1,v_2,\cdots,v_n\}$, 則

$L(G)=[l_{ij}]$, 其中 $l_{ii}=\deg(v_i)$, $i=1,2,\cdots,n$;

$l_{ij}=-1$ if $v_i \sim_G v_j$, 其它位置為 0.

例:



$$
\begin{array}{c|c}
 & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\
\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} &
\begin{bmatrix}
2 & -1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
-1 & -1 & 3 & -1 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}
\end{array}
$$

(1) 每一行(每一列)的和皆為0.

(*) 利用矩陣的 Eigenvalues 來研究圖的特性是代數圖論的主軸; 至於如何把圖轉成矩陣比較好用, 就需要一些想像力。

(*) 我們也可以把上述的"−1"都改成"1", 是另一種表示法。

Theorem (Number of k-walks) (A walk with k edges is a k-walk.)

Let G be the graph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and A_G be its adjacency matrix. Then, the number of different k-walks from v_i to v_j is equal to $A_G^k(i,j)$.

Proof. By induction on k and it is clear for $k=1$.

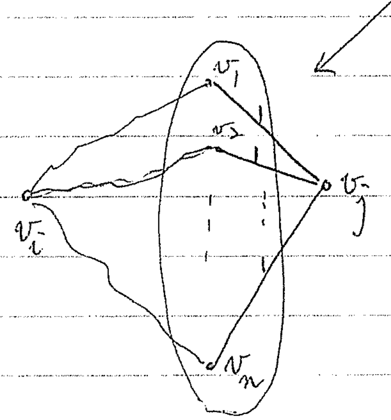
Assume the assertion is true for $1 \le h < k$. Consider the number of $(h+1)$-walks in $A_G^{h+1} = (A_G^h)A_G$. By multiplication of two matrices $A_G^{h+1}(i,j) = (A_G^h(i,1), A_G^h(i,2), \cdots, A_G^h(i,n)) \cdot$ Inner product

$(A_G(1,j), A_G(2,j), \cdots, A_G(n,j)) = \sum_{t=1}^{n} A_G^h(i,t) \cdot A_G(t,j)$. —— (1)

(transpose of the j^{th} column

Since the summation of (1) shows the number of $(h+1)$-walks from v_i to v_j, we have the proof.

Through $v_1, v_2, \cdots, v_n$ respectively



Facts

1. If $\deg_G(v_t) = 1$, then $A_G^{2m+1}(t,t) = 0$ for $m = 0, 1, 2, \cdots$.

2. The number of triangles in G is equal to $c_3(G) = \frac{1}{6}\sum_{i=1}^{n} A_G^3(i,i) = \frac{1}{6}\,tr(A^3)$. (trace)

3. For any two vertices v_i and v_j in $V(G)$, if there exists a k such that $A_G^k(i,j) > 0$, then G is a connected graph.

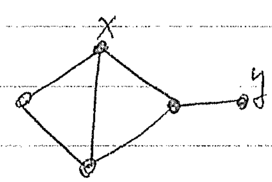
4. The number of 4-cycles in G is equal to $c_4(G) = \frac{1}{8}\left(tr(A^4) - 2\|G\| - 2\sum_{i\neq j} A^2(i,j)\right)$.

5. More $\cdots$ $c_5, c_6, c_7, c_8, \cdots$.

Review

1. The length of a path (or cycle, walk, trail, circuit) is the number of edges on the path (or ...).

2. The distance of two vertices x and y in a graph, denoted $dist(x,y)$, is the length of a shortest path between x and y. (min. # of edges)



$dist(x,y) = 2$.

Distance in Graphs (Continued)

Let $d(x,y)$ denote the distance of x and y in G.
($\mathrm{dist}(x,y)$ or $\mathrm{dist}_G(x,y)$) (Distance is a "metric".)

Notations

1. $e_G(x) = \max\{d(x,y) \mid y \in V(G)\}$ (Eccentricity of x in G)

2. $\mathrm{rad}(G) = \min\{e_G(x) \mid x \in V(G)\}$ (Radius of G)

3. $\mathrm{diam}(G) = \max\{e_G(x) \mid x \in V(G)\}$ (Diameter of G)
$$\| $$
$$\max\{d(x,y) \mid x, y \in V(G)\}$$

4. x is a central vertex of G if $e_G(x) = \mathrm{rad}(G)$.

5. $Z(G) = \{x \mid x \text{ is a central vertex}\}$.
(Center of G)

<u>Proposition</u> If G is a connected graph, then $\mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$.

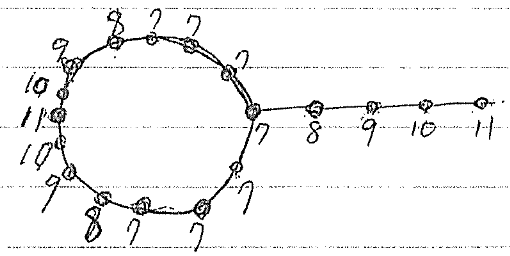
<u>Proof.</u> Let x and y be two vertices in G such that $d(x,y) = \mathrm{diam}(G)$. Let $w \in Z(G)$, i.e., $e_G(w) = \mathrm{rad}(G)$. Since

$$d(x,y) \le d(x,w) + d(w,y) \le e_G(w) + e_G(w) = 2\,\mathrm{rad}(G). \quad \blacksquare$$
(Triangular inequality)

(*) $\mathrm{rad}(G) \le \mathrm{diam}(G)$ is easy to see.

<u>Fact</u> For each pair of positive integers a and b such that $a \le b \le 2a$, there exists a graph G with $rad(G) = a$ and $diam(G) = b$.

For example, $a = 7$ and $b = 11$. (A cycle of length $2a+1$ with a "tail" P_{b-a+1}.)

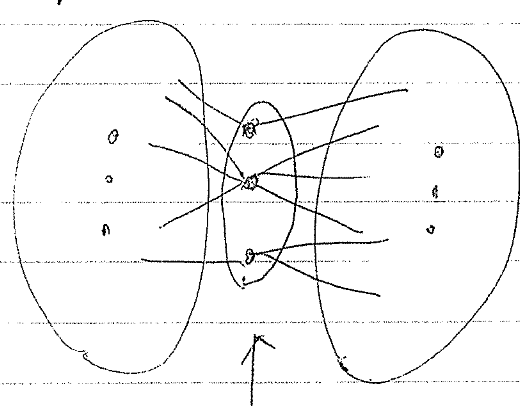


(*) $a = 7$, $b = 15$ is not possible, why?

<u>Exercise</u> Find a <u>3-connected</u> graph G such that

$rad(G) = 7$ and $diam(G) = 11$.

<u>Review</u> A graph G is k-connected if any separating set is of size at least 3.



separating set

For reference about Metric Space

Definition (Metric space)

A metric space (defined on M) is an ordered pair (M, d) where M is a set and d is a metric on M, i.e., a function $d : M \times M \to \mathbb{R}$ such that for any $x, y, z \in M$, we have

1. $d(x,y) = 0 \Leftrightarrow x = y$; (identity of indiscernibles)
2. $d(x,y) = d(y,x)$; and (symmetry)
3. $d(x,z) \le d(x,y) + d(y,z)$. (triangle inequality)

Note $d(x,y) \ge 0$ (non-negativity or separation axiom)

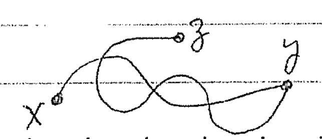
Proof. $2\,d(x,y) = d(x,y) + d(y,x) \ge d(x,x) = 0$. $\blacksquare$

Fact The "distance" defined on graph G, d, is a metric.

⓪ $d : V(G) \times V(G) \to \mathbb{R}$

① $d(x,y) = 0$ iff $x = y$, ② $d(x,y) = d(y,x)$, and

③ $d(x,y) + d(y,z) \ge d(x,z)$. (Why?)



If there exists a walk from x to z, then there exists a path from x to z, and the length of path is not larger than that of the walk

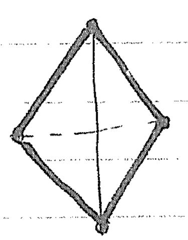
Hamilton Cycles (hamiltonian cycles)

Definition (Hamilton cycles)

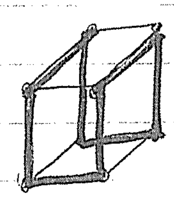
A cycle in G is a Hamilton cycle if the cycle contains all vertices of G.

Problem (Thomas P. Kirkman) 1855

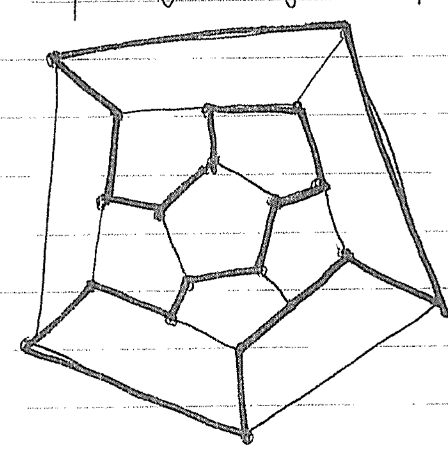
Given ~~the~~ graph of a polyhedron, can one always find a circuit (we call cycle) that passes through each vertex once and only once? (Yes for regular polyhedron!)



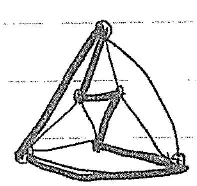
(tetrahedron)



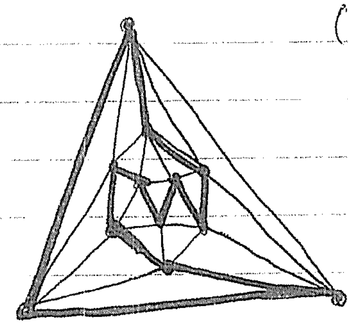
(Cube)



(dodecahedron)
(World Game!)
20 cities



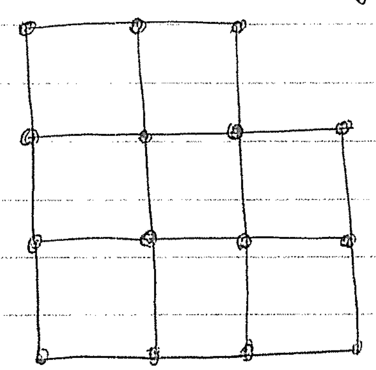
(octahedron)



(icosahedron)

From that time and on, determining whether a graph contains a Hamilton cycle becomes one of the most interested and important problem in graph theory.

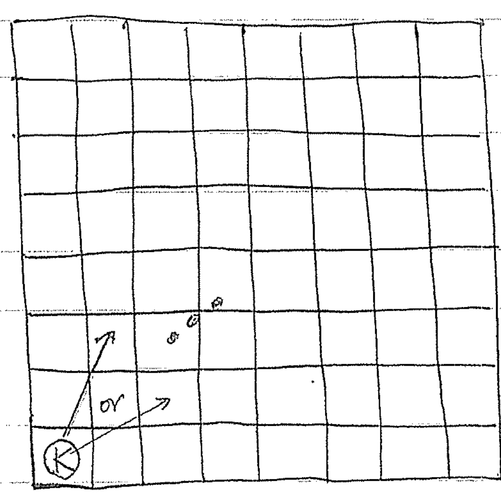
Test Can you find a Hamilton cycle in the following graph?



No! Why?

If a bipartite graph has a Hamilton cycle, then the graph must have even number of vertices.

Test Can you find a "Knight" in a chessboard which jumps all the 64 spots?



See 2-2'

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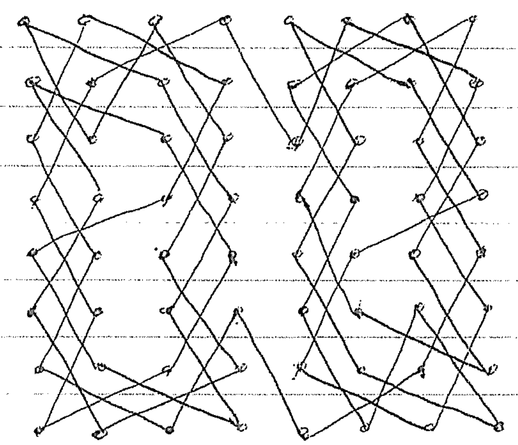

One tour of a knight on a chessboard