FGT Lecture 3 Sep. 27 - Oct. 4

Eulerian circuits

<u>Note</u> The first paper of Graph Theory deals with the existence of Eulerian circuits (eulerian circuits).

Walk : A sequence of vertices in G $\langle x_1, x_2, \cdots, x_n \rangle$ such that $x_i x_{i+1}$ is an edge of G where $i = 1, 2, \cdots, n-1$.

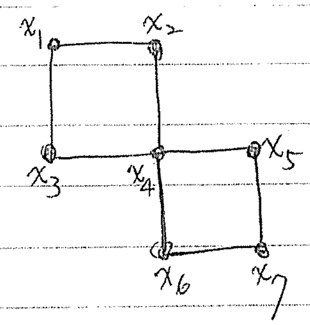
Trail : A walk without repeating edges

Circuit : A closed trail, i.e., the first vertex is also the last vertex.

<u>Cycle</u> : A circuit without repeating vertices. $(C_n) \Rightarrow \|C_n\| = n$.

<u>Path</u> : A trail (open) without repeating vertices. $(P_n) \Rightarrow \|P_n\| = n-1$.

<u>Definition</u> An <u>eulerian</u> circuit of a graph G (multi-graph) is a circuit of G which contains all edges of G.

G: 

$\langle x_1, x_2, x_4, x_5, x_7, x_6, x_4, x_3, x_1 \rangle$

$= (x_1, x_2, x_4, x_5, x_7, x_6, x_4, x_3)$ is an eulerian circuit of G.

SEASON

(*) A graph is connected if for any two vertices x and y there exists a <u>walk</u> (path) from x to y. (可以用 path 取代？)

Theorem (Euler, 1736 (1741))

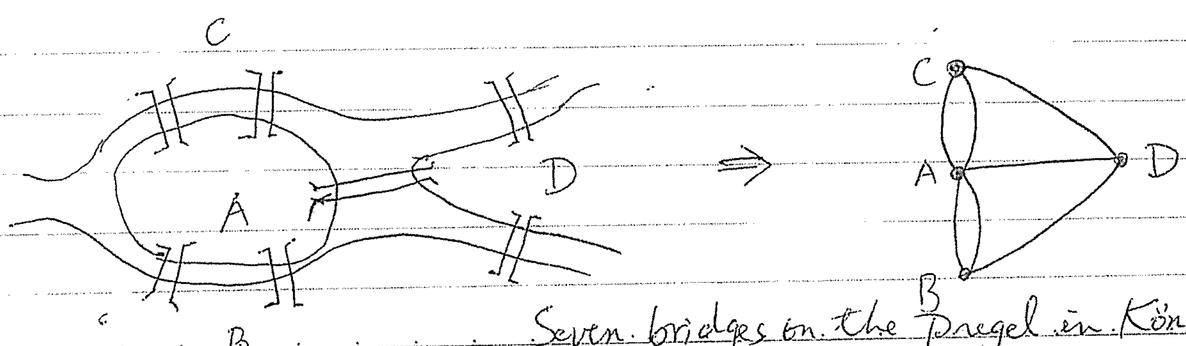
A graph G contains an eulerian circuit if and only if G is connected and every vertex is of even degree.

(Remark: The graph considered by Euler is a multi-graph.)

Proof. (Many versions) ($\Rightarrow$) is easy to see, we prove ($\Leftarrow$).

1st proof: (Best?) Let $T = \langle x_1, x_2, \ldots, x_\ell \rangle$ be the longest trail in G starting from x_1 and ending at x_ℓ. Since each vertex is an even vertex, $x_1 = x_\ell$. Let E' be the set of edges of this trail. If $\|E'\| = \|G\|$, then the trail is an eulerian circuit. Assume that $E'' = E(G) \setminus E' \neq \phi$.

Since G is connected, $V(E'') \cap V(T) \neq \phi$. Let x_i be a common vertex and $x_i x_k \in E''$. Now, we have a longer trail starting from x_k, $\langle x_k, x_i, x_{i+1}, \ldots, x_\ell (= x_1), x_2, \ldots, x_i \rangle$, a contradiction. Hence $E'' = \phi$ and T is an eulerian circuit. ∎



Seven bridges on the Pregel in Königsberg

2nd proof. (The main idea of Euler) By induction on $\|G\|$.

Since each vertex of G is even, $\delta(G) \geq 2$, G contains a cycle. Let C be a circuit in G with the maximal number of edges. (?)
If $\|C\| = \|G\|$, then C is an eulerian circuit of G. On the other hand,

$E(G) \setminus E(C) \neq \phi$, i.e, C is not an eulerian circuit of G. As G is connected,

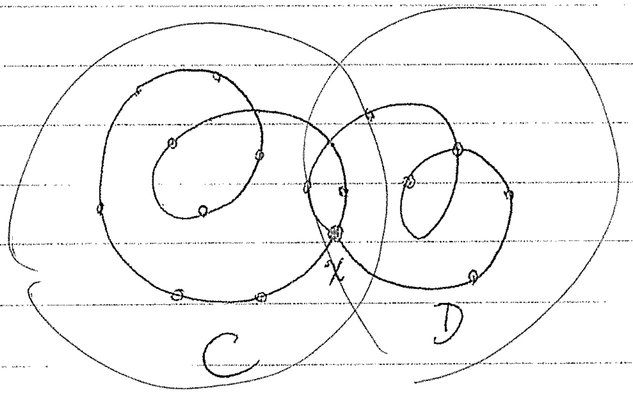
C contains a vertex x in a non-trivial component H of $G - E(C)$.

By induction H has an eulerian circuit D. Since $E(C) \cap E(D) = \phi$

and $V(C) \cap V(D) \supseteq \{x\}$, we obtain a larger circuit of G by

concatenating these two circuits. Hence C is an eulerian

circuit of G. ▨



Remark At the time of "Euler", mathematical induction

is not known yet.

Different proofs ...?

SEASON

Lemma

In a graph G, if $\delta(G)$ (minimum degree) ≥ 2, then G contains a cycle of length at least $\delta(G)+1$.

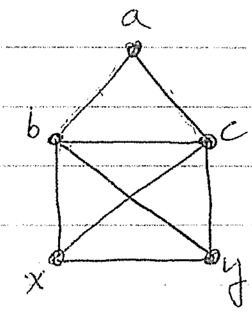
Proof. Let $\langle x_1, x_2, \cdots, x_\ell \rangle$ be a path of maximum distance from x_1. Then, $N_G(x_1) \subseteq \{x_2, x_3, \cdots, x_\ell\}$. Let x_t be the vertex with largest index. Clearly, $t \geq \delta(G)+1$. This concludes the proof.

(o) We also conclude that $\ell \geq \delta(G)+1$, hence G contains a path of length at least $\delta(G)$.

3rd proof of Eulerian circuit theorem

$\boxed{\text{By induction of } \|G\|.}$ It is true if $\|G\|=2$. Assume that the assertion is true for G' with $\|G'\| < \|G\|$ and G' satisfies the sufficient conditions, G' is connected and every vertex is of even degree. Since $\delta(G) \geq 2$, G contains a cycle Z. Then, $G-Z$ is either an empty graph or a graph of smaller size, such that each component of $G-Z$ satisfies the S.C., Now, by induction and the idea of Euler, we have the proof. (put them together!) ▨

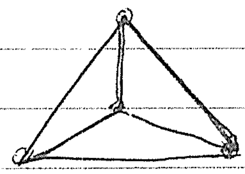
Corollary A connected graph has <u>an eulerian trail</u> (一筆畫) from a vertex x to a vertex $y \neq x$ if and only if x and y are the only odd vertices.



Eulerian Trail : $\langle x, b, a, c, x, y, b, c, y \rangle$

Corollary If a connected graph G has $2m$ ($m \geq 1$) odd vertices, then the edges of G can be partitioned <u>into m trails</u> (1st part) such that <u>each trail has an even number of edges except possibly one.</u> (2nd part)

(No easy to prove! Especially the second part.)



<u>Digraph version</u> (To be continued!)

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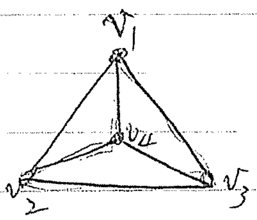
Proof of the first part.

Let the $2m$ odd vertices of G be $\{x_1, x_2, \cdots, x_{2m-1}, x_{2m}\}$ and $V(G) = \{x_1, x_2, \cdots, x_p\}$. Clearly $p \geq m$. Now, consider the graph $\tilde{G}$ with $V(\tilde{G}) = \{x_0, x_1, x_2, \cdots, x_p\}$ and $E(\tilde{G}) = E(G) \cup \{x_0 x_i \mid i = 1, 2, \cdots, 2m\}$. Then, $\tilde{G}$ is a connected even graph and thus $\tilde{G}$ has an eulerian circuit. Since the circuit passes x_0 exactly $2m$ times if we start at a vertex which is not x_0, $\tilde{G} - x_0$ is decomposed into m trails which concludes the proof of the first part.

An example of 2nd part



$\langle v_1, v_4, v_2, v_3 \rangle$ and $\langle v_2, v_1, v_3, v_4 \rangle$ are two trails (Both have length 3)

$\Downarrow$

$\langle v_1, v_4, v_2, v_1, v_3 \rangle$ and $\langle v_4, v_3, v_2 \rangle$

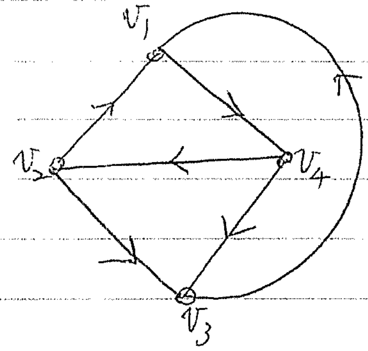
are two even trails

SEASON

Digraph (Directed Graph)

<u>Definition</u> A digraph $D = (V, A)$ is an ordered pair such that V is the set of vertices and A is a set of <u>ordered pairs</u> (arcs) in (V, V) or V^2. $|A|$ is defined as the size of D (the number of arcs).

e.g.

D : 

$\deg^+_D(v_1) = 1$, $\deg^-_D(v_1) = 2$
$\deg^+_D(v_2) = 2$, ⋮
$\deg^+_D(v_3) = 1$,
$\deg^+_D(v_4) = 2$,

$$V(D) = \{v_1, v_2, v_3, v_4\}$$

$$A(D) = \{(v_1, v_4), (v_2, v_1), (v_2, v_3), (v_3, v_1), (v_4, v_2), (v_4, v_3)\}$$

$N^+_D(v) = \{u \in V \mid (v, u) \in A\}$, $N^-_D(v) = \{u \in V \mid (u, v) \in A\}$

↑ out-neighbor ↑ in-neighbor

$\deg^+(v) = |N^+_D(v)|$, $\deg^-(v) = |N^-_D(v)|$

↑ out-degree of v ↑ in-degree of v

<u>Theorem</u> Let $D = (V, A)$ be a digraph. Then

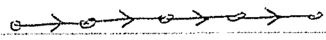
$$\sum_{v \in V(D)} \deg^+_D(v) = \sum_{v \in V(D)} \deg^-_D(v) = |A|.$$

Remark Walks, trails, circuits, cycles and paths in digraph D are defined according the directions, i.e., in a walk $\ll v_1, v_2, v_3, \cdots \gg$ $(v_1, v_2), (v_2, v_3)$ are arcs in D. For clearness, we shall say a directed path in D instead of just a path in D.

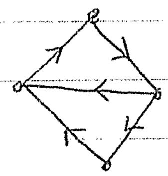
Definition (Connected digraphs)

A digraph D is connected if for any two distinct vertices x and y in V(D), there exists a directed path from x to y or from y to x. If we replace "or" by "and", then the digraph is strongly connected.

weakly connected



weakly connected



strongly connected

Directed eulerian circuits

<u>Theorem</u> A digraph D has an eulerian (directed) circuit if and only if D is strongly connected and for each vertex $v \in V(D)$, $\deg^+(v) = \deg^-(v)$.

<u>Proof</u>. Similarly we can apply the idea used in graphs. ■

In fact, we can count the number of distinct directed eulerian circuits in an eulerian digraph. (A digraph with eulerian circuits is called an eulerian digraph.)

<u>Theorem</u> (BEST Theorem) $(V(D) = \{v_1, v_2, \cdots, v_n\})$.

Let D be an eulerian digraph, $s(D)$ be the number of eulerian circuits, and $t_i(D)$ be the number of spanning trees oriented from v_i. Then $s(D) = t_i(D) \cdot \prod_{j=1}^{n} (d^+(v_j) - 1)!$ for every i, $1 \le i \le n$. (Note that $t_1(D) = t_2(D) = \cdots = t_n(D)$)

(The theorem was proved by de Bruijn, van Aardenne-Ehrenfest, Smith and Tutte.) <u>We skip the details.</u>

SEASON

Remark

Counting the number of distinct eulerian circuits in an eulerian graph (not directed) is a very difficult problem. (Can you do it?)

There are many applications of "eulerian circuits", please refer to lecture notes "Chapter 2". For convenience, I attach them following this page.

SEASON

2. De Bruijn 數列及郵差問題

這一節, 我們將討論兩個尤拉圖概念的應用。

定義 2.1. (de Bruijn Sequence)

一個循環數列 $(a_1, a_2, ..., a_{2^n})$ 稱爲是 $(2,n)$- de Bruijn 數列如果下列兩個條件滿足:

(1) $a_i \in \{0,1\}$, $i = 1, 2, ..., 2^n$; 且

(2) $(a_j, a_{j+1}, \ldots, a_{j+n-1})$, $j = 1, 2, \ldots, 2^n$, $(\text{mod } 2^n)$, 爲相異的 2^n 個 n 維向量。

例1. $n = 1, 2, 3, 5$; $(2,n)$-de Bruijn 數列分別爲 (括號, 逗號省略)

01, 0110, 01110100, 0000100110101111 。

對於所有的 n 要建構一個 $(2,n)$- de Bruijn 數列並不困難, 以下是荷蘭數學家 N. de Bruijn 用來尋找這種 數列的有向圖。

定義 2.2. ($(2,n)$-de Bruijn 有向圖, $D_{2,n}$)

$D_{2,n}$ 爲一加權有向圖, 它滿足下列兩條件:

(1) $V(D_{2,n}) = (Z_2)^{n-1}$ 及

(2) $(a_1, a_2, \ldots, a_{n-1})$ 連到 $(a_2, a_3, \ldots, a_{n-1}, a_n)$ 並且在這個弧上給予加權 $(a_1, a_2, \ldots, a_n)$ 。

例2.

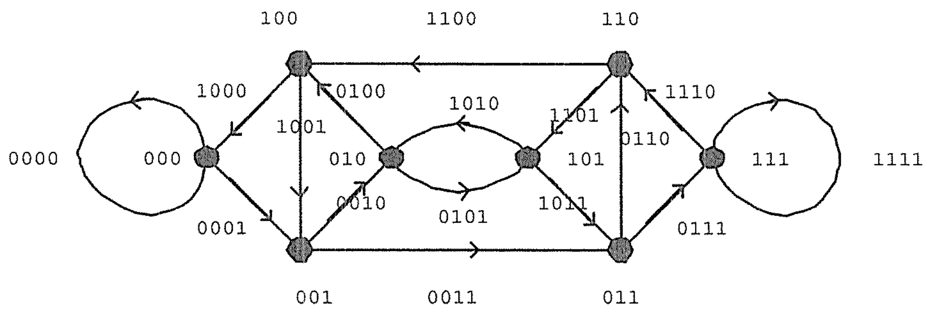


圖 5 : $D_{2,4}$

由定義, 我們可以證明 $D_{2,n}$ 爲一有向的尤拉圖。

引理 2.3. $(2,n)$-de Bruijn 有向圖爲尤拉圖。

證明.

因爲 $D_{2,n}$ 爲強連通圖而且每一點的內度與外度均爲 2, 故得證。 ■

引理 2.4. 在 $(2,n)$-de Bruijn 有向圖上的加權全部不一樣。

證明.

由定義可得。 ■

定理 2.5. 對於所有的 n , $(2,n)$-de Bruijn 數列存在。

證明.

首先, 由引理 2.3, 一個 $(2,n)$-de Bruijn 有向圖存在, 所以存在一個尤拉迴路 ; 現在, 令此迴路經過的邊爲 $e_1, e_2, \ldots, e_{2^n}$; 同時對於所有的 i 令 $l(e_i)=a_i$ 爲 e_i 邊上加權的 最左邊那個數字, 於是我們得一個數列 $(a_1, a_2, \ldots, a_{2^n})$, 這個數列就是一個 $(2,n)$-de Bruijn 數列。(作業6.) ■

以下的例子也許有助於了解上述的證明。

在$D_{2,4}$中的尤拉迴路可以是

$$
\begin{aligned}
&000 \xrightarrow{0000} 000 \xrightarrow{0001} 001 \xrightarrow{0010} 010 \xrightarrow{0100} 100 \xrightarrow{1001} 001 \xrightarrow{0011} 011 \\
&\xrightarrow{0110} 110 \xrightarrow{1101} 101 \xrightarrow{1010} 010 \xrightarrow{0101} 101 \xrightarrow{1011} 011 \xrightarrow{0111} 111 \\
&\xrightarrow{1111} 111 \xrightarrow{1110} 110 \xrightarrow{1100} 100 \xrightarrow{1000} (000)
\end{aligned}
$$

(2,4)-de Bruijn 數列爲 0000100110101111 , 而它所產生的 16 個不同之向量依次爲 16 個加權向量。

De Bruijn 數列最出名的應用是在旋轉鼓 (Rotating Drum) 。它可以利用連續的位置來判斷不同的輸入 (機械原理) , 如下圖所示。

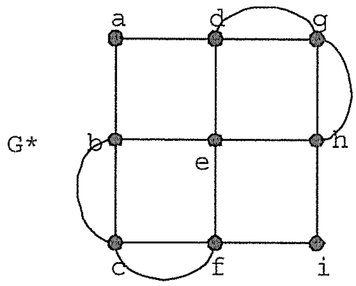


圖 7 : 最佳郵差路線

演算法 (Edmonds **及** Johnson) (**摘要**)

1. 令 S 爲 G 中奇點所成的集合。

2. 利用 S 的點建構一個加權完全圖 $K_{|S|}$, ab 邊上的加權爲由 a 到 b 的最短距離 (加權路徑) 。

3. 在 $K_{|S|}$ 中選出 $\frac{|S|}{2}$ 個獨立邊, 使得它們的加權總和爲最小。

4. 對應於 3 中的獨立邊, 將所經過的邊 (最短路徑) 全部加上相同加權的邊。

5. 由 4 所得到的圖之尤拉迴路即爲所求。

證明.
3 的找法是多項式時間, 所以這個演算法也是, 詳細證明在此省略。■

其它方面上有不少應用, 例如在掃街路線的安排, 要怎樣行駛才最省經費 ; 在 RNA 的重組方面如何利用片斷資 訊來得到完整的 RNA ; 以及在資訊傳遞方面如何編碼才更有效率地把資料表現出來, 都可以利用有向的尤拉圖 來達到目的 ; 由於篇富有限在此省略, 請自行參考由 Gross 及 Yellen 所寫的書 "Graph Theory and its Applocation"。

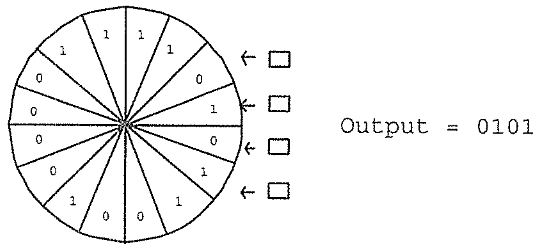


圖 6 : 旋轉鼓

另一個出名的應用是由華裔數學家管梅谷 (Meigu Guan) 所提出來, 後來 J. Edmonds 稱它為中國郵差問題。

定義 2.6. (郵差路線 , Postman Tour)

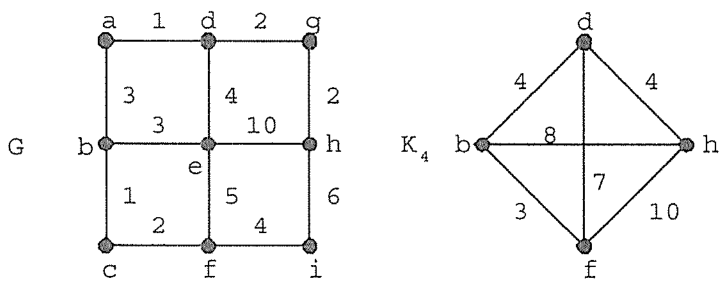
任給一個圖 G 所謂郵差路線是指在 G 中的一個封閉步行 (Closed Walk) , 它通過 G 中的每一個邊至少一次。

定義 2.7. (最佳的郵差路線 , Optimal Postman Tour)

在一個加權圖中, 一個郵差路線所經過的邊加權總和為最小時, 我們稱這路線為最佳郵差路線。

這個問題看來十分複雜。因為所考慮的圖千萬化, 然而, 在提出這個問題 (尋找最佳郵差路線) 之後不到十年, 即 由 Edmonds 和 Johnson 所解決, 他們提供了一個多項式時間 (Polynomial-time) 的演算法來找出最佳路線。

在沒有介紹演算法之前, 我們先看一個例子。在 G 中有 4 個奇點, 現在利用這 4 點建構一個加權的 K_4 , 在邊上的加權為兩點在 G 中的最短距離 (加權) 。接著在 K_4 中找到不相鄰的兩邊 (配對) 使得



它們的加權和最小, 此例中的 bf 即 dh , 最後再將原圖 G 改為 G^* , 如圖7, G^* 的尤拉迴 路即為所求。