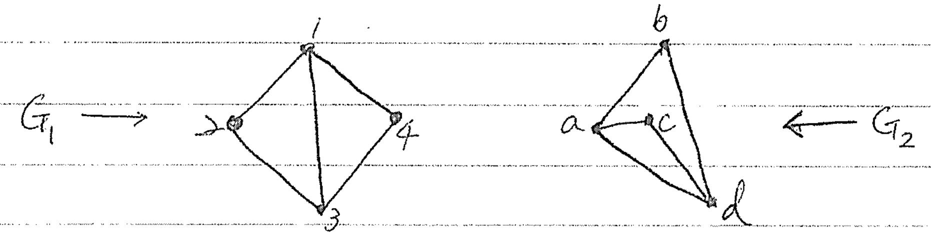
FGT Lecture 2 Sep. 20-24

Graph Isomorphism for Simple Graphs

Given two graphs, for example the following:



Are they "the same" or not ?

<u>Fact</u> If $V(G_1) \neq V(G_2)$ or $E(G_1) \neq E(G_2)$, then $G_1 \neq G_2$.

<u>Observation</u>

(1) G_1 and G_2 are of the same order, i.e., $|G_1| = |G_2|$.

(2) G_1 and G_2 are of the same size, i.e., $\|G_1\| = \|G_2\|$.

(3) G_1 and G_2 are of the same "incidence" structure:

$1 \to a$, $3 \to d$, $2 \to c$, $4 \to b$.

<u>Review</u>: Two sets A and B are of the same <u>cardinality</u> if there exists a bijection between A and B, i.e., $\exists$ $f: A \xrightarrow[\text{onto}]{1-1} B$. ($A$ and B can be infinite sets.)

Definition (Isomorphic Graphs)

Two graphs G_1 and G_2 are isomorphic if there exists a bijection $f: V(G_1) \longrightarrow V(G_2)$ such that

$$\underset{\substack{\parallel\!\!\!| \\ uv \in E(G_1)}}{u \sim_{G_1} v} \iff \underset{\substack{\parallel\!\!\!| \\ f(u)f(v) \in E(G_2)}}{f(u) \sim_{G_2} f(v)}.$$

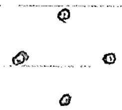
We use $G_1 \cong G_2$ to denote the fact that G_1 is isomorphic to G_2.

Fact $\quad G_1 \cong G_2 \Rightarrow \begin{cases} ① \ |G_1| = |G_2| \\ ② \ \|G_1\| = \|G_2\| \\ ③ \ \text{Many others} \cdots. \\ \quad (\text{properties}) \end{cases}$

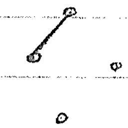
Fact Two graphs are non-isomorphic if they are not isomorphic.

Fact Isomorphism is an equivalence relation defined on $\mathcal{G} \times \mathcal{G}$ where $\mathcal{G}$ is the set of all simple graphs. (graphs)

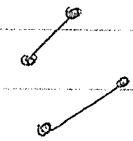
Example: Non-isomorphic graphs of order 4 (simple)



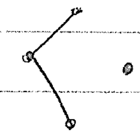
O_4



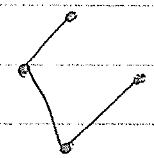
$O_2 \cup K_2$



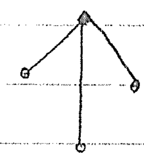
M_2



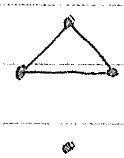
$P_3 \cup O_1$



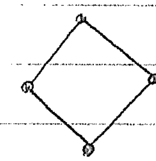
P_4



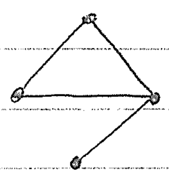
S_3

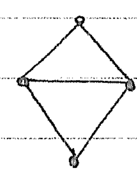


$K_3 \cup O_1$ or $C_3 \cup O_1$

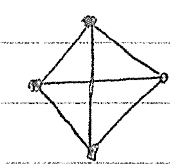


C_4

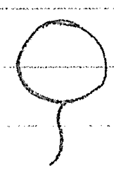




$K_4 - e$



K_4

(

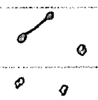
Tadpole)

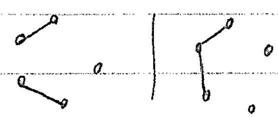
Homework: (How many non-isomorphic graphs are of order 6?)

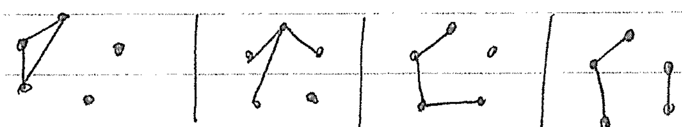
There are 34 non-isomorphic graphs of order 5.
(simple)

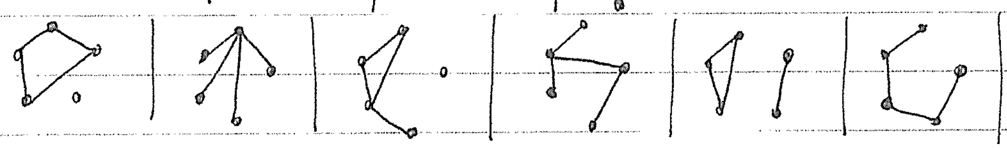
<u>Idea</u> First, we use the number of edges to partition the set of graphs of order 5.

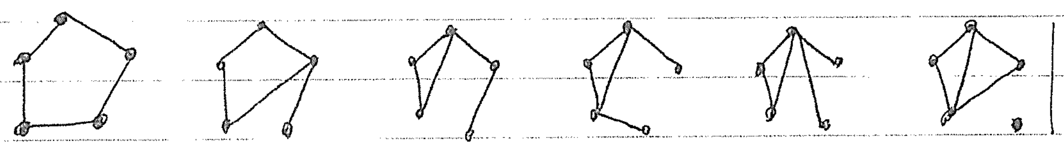
1. $e=0$,  1

2. $e=1$  1

3. $e=2$  2

4. $e=3$  4

5. $e=4$  6

6. $e=5$  6

7. $e=6 \approx e=4$ 6

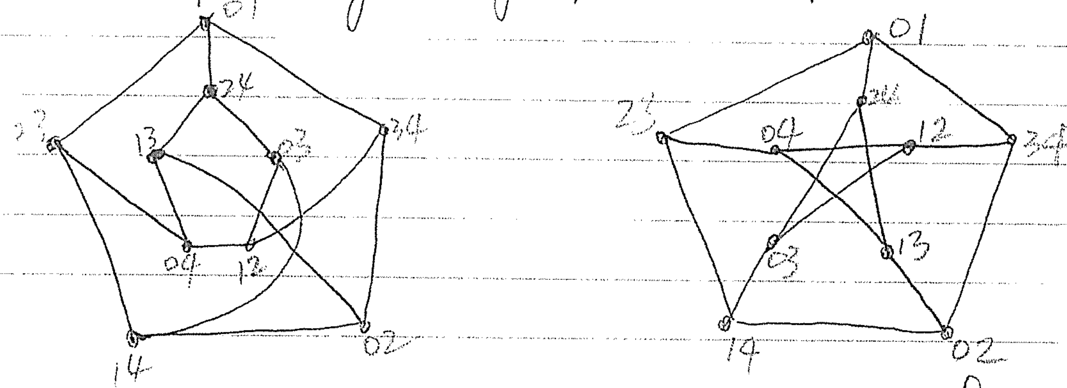
8. $e=7 \approx e=3$ 4

9. $e=8 \approx e=2$ 2

10. $e=9 \approx e=1$ 1

11. $e=10 \approx e=0$ 1

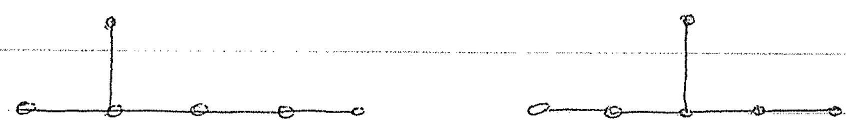
Are the following two graphs isomorphic?



Yes!

Petersen Graph

Are the following two graphs isomorphic?



No!

Fact Let $V = \binom{\mathbb{Z}_5}{2} = \{$ 2-element subsets of $\mathbb{Z}_5 \}$

$= \{01, 02, 03, 04, 12, 13, 14, 23, 24, 34\}$

and $E = \{ AB \mid A, B \in V \text{ and } A \cap B = \phi \}$. The (V, E) is isomorphic to Petersen graph. (Kneser Graphs!)

Famous Graphs

(!) So, how to determine whether two graphs are isomorphic or not. (Isomorphism Disease)

<u>Possible approach</u> (For reference, 参考用)

<u>Definition</u> (Degree sequence)

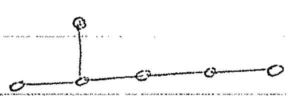
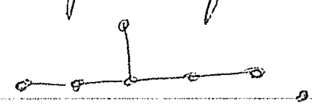
Let G be a graph and $V(G) = \{v_i \mid i = 1, 2, \cdots, n\}$. Let $d_i = \deg_G(v_i)$ and $\overset{\Delta(G)}{\overset{\parallel}{d_1}} \ge d_2 \ge \cdots \ge \overset{\delta(G)}{\overset{\parallel}{d_n}}$. Then, $\boxed{\langle d_1, d_2, \cdots, d_n \rangle \text{ is the degree sequence of } G.}$

<u>Review</u> 1. $\forall\ v \in V(G)$, $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. They are the neighborhood of v and closed neighborhood of v respectively.

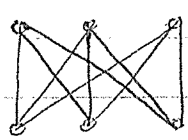
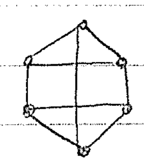
2. $|N(v)|$ is known as the $\boxed{\text{degree}}$ of v (denoted by $\deg_G(v)$). If the graph has a loop passing v, then this loop contributes two degrees.

3. A vertex is even (or odd) if its degree is even (or odd).

<u>Example</u> 1. $\langle 3, 2, 2, 1, 1, 1 \rangle$ is the degree sequence of both

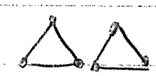
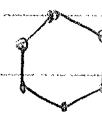
 and .

2. $\langle 3, 3, 3, 3, 3, 3 \rangle$ is the degree sequence of both

 and .

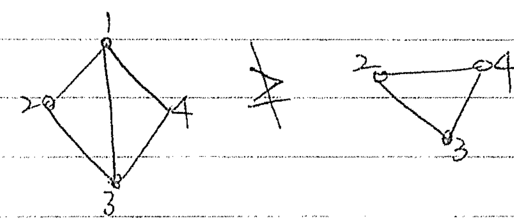
Can we say something about isomorphic graphs and degree sequences?

<u>Conclusion</u> (Trivial)

Two graphs may have the same degree sequence, but they are not isomorphic. ( , )

(Fact) If two graphs are isomorphic, then they have the same degree sequence.

Subgraphs

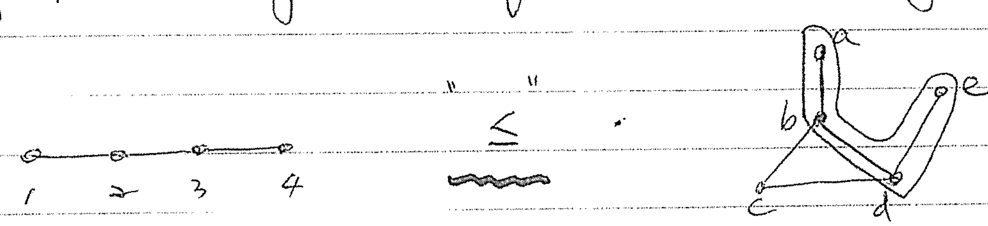


<u>Definition</u> (Strong sense)

Let G and H be two graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Then H is a subgraph of G.

<u>Definition</u> (General version)

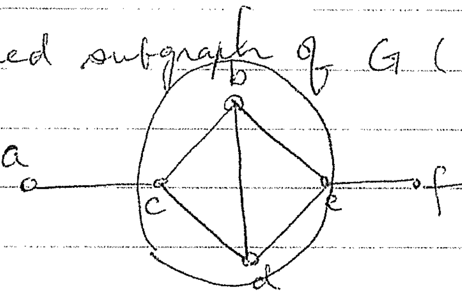
<u>H is a subgraph of G</u> if H is isomorphic to a subgraph (strong sense) of G, denoted by $H \leq G$.



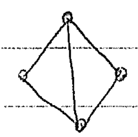
Definition (Induced subgraphs).

Let G be a graph and $S \subseteq V(G)$ be a non-empty subset. The graph induced (or generated) by S, $\langle S \rangle_G$, is a subgraph of G such that, for any two vertices u and v in S, $u \sim_{\langle S \rangle_G} v$ if and only if $u \sim_G v$. $\langle S \rangle_G$ is called an induced subgraph of G (induced by S).

G:



Let $S = \{b, c, d, e\}$. $\langle S \rangle_G$:



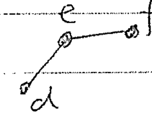
.

$\langle S \rangle_G \leq G$.
↑ induced subgraph

Definition (Edge-induced subgraphs)

Let G be a graph and $\tilde{E} \subseteq E(G)$. Then the edge-induced subgraph of $\tilde{E}$, $\langle \tilde{E} \rangle_G$, is the subgraph of G whose edge set is $\tilde{E}$ and vertex-set is the set of vertices in G which are incident to an edge in $\tilde{E}$.

For example, let $\tilde{E} = \{de, ef\}$, then $\langle \tilde{E} \rangle_G$:



.

$\boxed{\text{(Fact)}}$ If two graphs G_1 and G_2 are isomorphic, then H is a subgraph of G_1 if and only if H is a subgraph of G_2.

Definition (H-free graphs).

A graph G is said to be H-free if H is not a subgraph of G.

Example. Petersen graph is C_3-free and also C_4-free.

Review. A cycle with $n \geq 3$ vertices is denoted by C_n.

Definition (Extremal graphs)

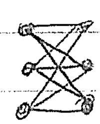
Consider all graphs $(\mathbb{G}_n)$ on n vertices. Let

$ex(n; H) = \max\{\|G\| \mid G \in \mathbb{G}_n \text{ and } G \text{ is } H\text{-free}\}$. The graphs G of order n and size $ex(n;H)$ are known as extremal graphs which forbids H.

Example. $n = 6$, $ex(6; C_3) = 9$. (why?)

Note Two steps to show.

Step 1. There exists a graph of order 6 and size 9 which is C_3-free.



$(ex(6; C_3) \geq 9)$

Step 2. All graphs of order 6 and size "≥ 10" contain a C_3.
(Harder step!) $(ex(6; C_3) \leq 9)$

(*) Prove the theorem

Theorem (Mantel, 1907) (Good to know!)
Every graph of order n and size greater than $\lfloor \frac{n^2}{4} \rfloor$ contains a K_3.

Since $K_3 \nsubseteq G$, for every $xy \in E(G)$, $N_G(x) \cap N_G(y) = \phi$. This implies that $\deg_G(x) + \deg_G(y) \le |G| = n$. (Figure 1) Now, consider

$$\sum_{xy \in E(G)} (\deg_G(x) + \deg_G(y)) = \sum_{x \in V(G)} (\deg_G(x))^2 \quad \text{(Two-way counting)}.$$

$$\le n \cdot \|G\| = n \cdot e(G)$$

By Cauchy's inequality, $(2e(G))^2 = \left(\sum_{x \in V(G)} \deg_G(x)\right)^2 \le n \cdot \sum_{x \in V(G)} (\deg_G(x))^2$

$$\le n^2 \cdot e(G).$$

Hence, $e(G) \le \frac{n^2}{4}$. ∎

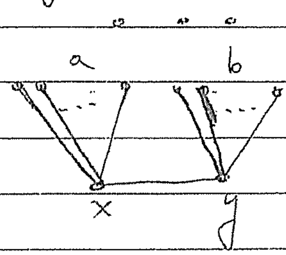


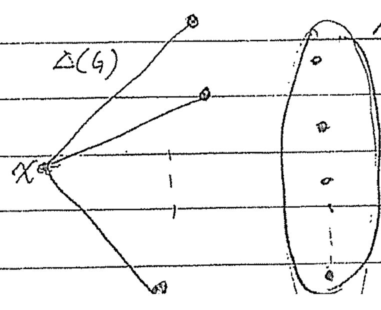
Figure 1. $\deg_G(x) = a+1$, $\deg_G(y) = b+1$

(2nd proof.)

Let $x \in V(G)$ be a major vertex, i.e., $\deg_G(x) = \Delta(G)$. (Figure 2)

Since $K_3 \nsubseteq G$, $\langle N_G(x) \rangle_G$ induces an empty graph. This implies that $\|G\| \le \Delta(G) + \Delta(G) \cdot (n - \Delta(G) - 1) = \Delta(G) \cdot (n - \Delta(G))$.

$\|G\|$ will take a maximum when $\Delta(G) = \lfloor \frac{n}{2} \rfloor$. Hence, we have the proof. ∎

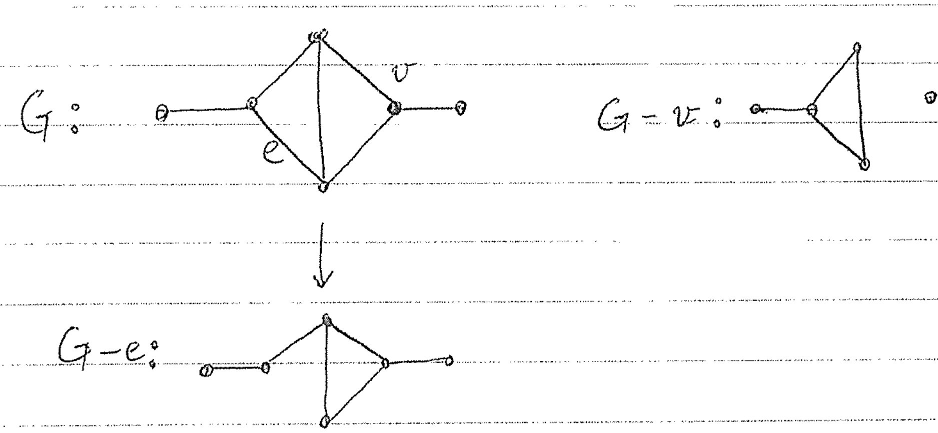


Problem $G \nsupseteq C_4$
$\|G\| = 14$
Find $\max\{\|G\|\}$.
(Bonus)

<u>Remark</u> To determine $ex(n;H)$ for a given H is a very difficult problem in general. But, for some special H, say $K_k, P_k, \cdots$, $ex(n;H)$ can be determined.

<u>Definition</u> (Deletion of vertices (and/or) edges)

Let $v \in V(G)$. The graph $G-v$ is obtained by deleting the vertex v and all the edges incident to v. The graph $G-e$ where $e \in E(G)$ is a subgraph of G with the deletion of e from $E(G)$.



Let $S \subseteq V(G)$ and $T \subseteq E(G)$. $G-S$ and $G-T$ can be defined accordingly. (Take away vertices (or edges) one by one.)

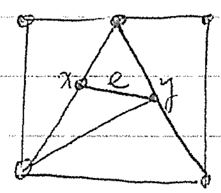
<u>Definition</u> (Graph minors)

A graph M is called a minor of G if M can be obtained from G by contracting edges, deleting vertices and edges.

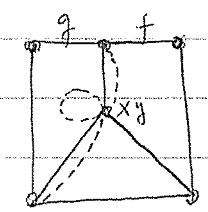
<u>Review</u> (Edge-contraction)

Given an edge $\overset{=e}{xy}$ of a graph G, the graph G/e is obtained from G by contracting e; that is to identify the vertices x and y and deleting resulting loops and duplicate edges.

G :



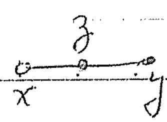
G/e :



<u>Example</u> K_4 is a minor of the above G.
(Contracting e, f and g.)

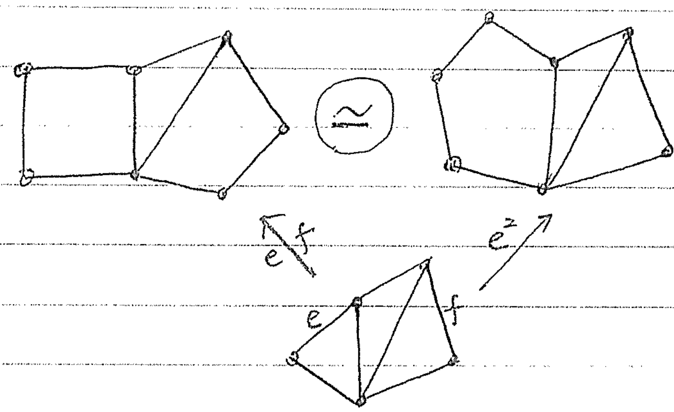
<u>Definition</u> (Subdivision)

A subdivision of an edge xy is obtained by adding a new vertex z such that we have edges xz and zy.



<u>Definition</u> (Homeomorphic Graphs)

Two graphs are homeomorphic if they can be obtained by subdividing edges (consecutively) of a fixed graph.



(These two graphs are "topologically" the same.)

<u>Remark</u>. Two cycles are homeomorphic.