Lecture 20 (For reference ③)

(•) A graph G has an $\mathcal{H}$-decomposition if $E(G)$ can be partitioned into subsets $E_1, E_2, \cdots, E_k$ such that for each $i = 1, 2, \cdots, k$, $\langle E_i \rangle_G \in \mathcal{H}$.

(•) If $\mathcal{H} = \{H\}$, then an $\mathcal{H}$-decomposition of G can be referred as an H-decomposition of G.

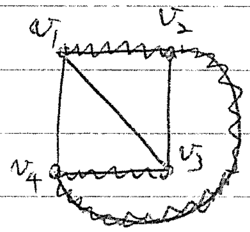
(•) A graph G has an $\mathcal{H}$-packing if $E(G)$ contains edge-disjoint subsets such that each of them induces a graph in $\mathcal{H}$. An H-packing of G can be defined accordingly.

(•) A graph G has an $\mathcal{H}$-covering if $E(G)$ is a subset of a edge-disjoint union of graphs in $\mathcal{H}$. An H-covering of G can be defined as well.

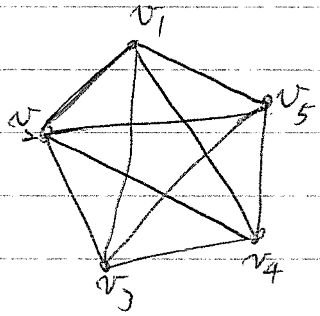
(••) In packing, the edges not used induce a subgraph which is known as the "leave" of the packing. Similarly in covering, the extra edges used induce a padding of the covering.

(◦) If the graph G is the complete graph K_n, then the H-decomposition, H-packing and H-covering is also referred to as that of order n.

Examples



$\Rightarrow$ P_4-decomposition
$\{\langle v_1, v_2, v_4, v_3\rangle, \langle v_2, v_3, v_1, v_4\rangle\}$



$\Rightarrow$ C_3-packing: $\{(v_1, v_4, v_5), (v_1, v_2, v_3)\}$
with leave a C_4, (v_2, v_3, v_4, v_5).

$\downarrow$

C_3-covering: $\{(v_1, v_4, v_5), (v_1, v_2, v_3), (v_2, v_4, v_4), (v_2, v_4, v_5)\}$

with a padding .

Theorem 96

For each odd integer $n \geq 3$, K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles. For each even integer n, K_n can be decomposed into $\frac{n}{2}$ Hamilton paths.

Proof. The following construction is known as the Waleki's method.

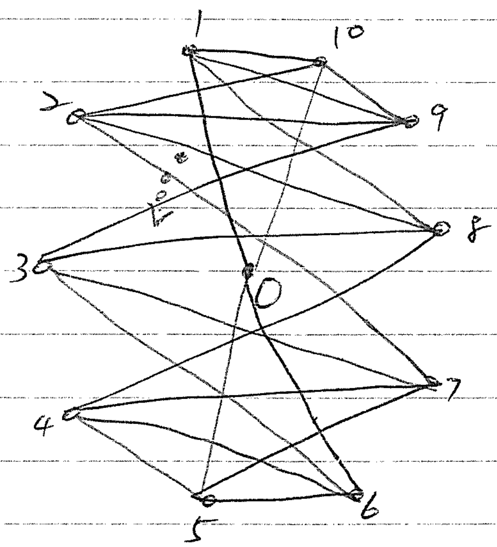
Let $V(K_n) = \mathbb{Z}_n$ and the cycles are :

$$(0, 1, n-1, 2, n-2, \dots, \tfrac{n-1}{2}, \tfrac{n+1}{2}),$$

$$(0, 2, 1, 3, n-1, \dots, \tfrac{n+1}{2}, \tfrac{n+3}{2}),$$

$$\vdots$$

$$(0, \tfrac{n-1}{2}, \tfrac{n-3}{2}, \tfrac{n+1}{2}, \tfrac{n-5}{2}, \dots, n-2, n-1).$$ ▨



By deleting a vertex in K_{n+1} (n even), we obtain the decomposition of K_n into $\frac{n}{2}$ Hamilton paths. ▨

(*) This theorem has been extended to cycles and paths with prescribed length. We list the theorem and omit their proofs. (The cycle case is very complicate.)

<u>Theorem 96'</u> (Alspach et. al, 2001)

For each odd integer larger than 3 and an integer $3 \le m \le n$, the complete graph K_n (n is odd) and $K_n - I$ (n is even) can be decomposed into m-cycles provided $m \mid \binom{n}{2}$ (for odd n) and $m \mid \binom{n}{2} - \frac{n}{2}$ (for even n) respectively.

(∘) The case $m=3$ was proved in 1847 by T.P. Kirkman, and the case $m=n$ mentioned above was obtained long time ago.

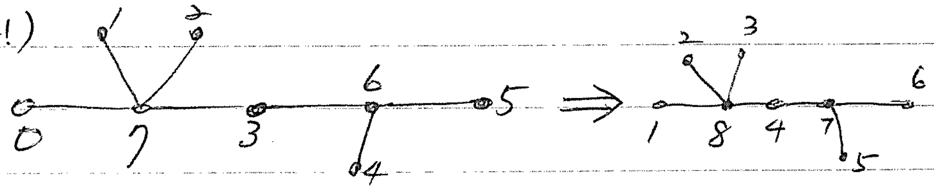
(*) An important tool for decomposition.

<u>Definition</u> (Graceful labeling, β-labeling)

A graceful labeling of a graph G is a 1-1 mapping $f : V(G) \longrightarrow \{0, 1, 2, \cdots, \underline{\max\{\|G\|, |G|-1\}}\}$ such that the weights of edges uv defined by $|f(u) - f(v)|$ are all distinct.

If G is connected, this <u>value</u> takes $\|G\|$.

<u>Example</u> (Shown earlier!)

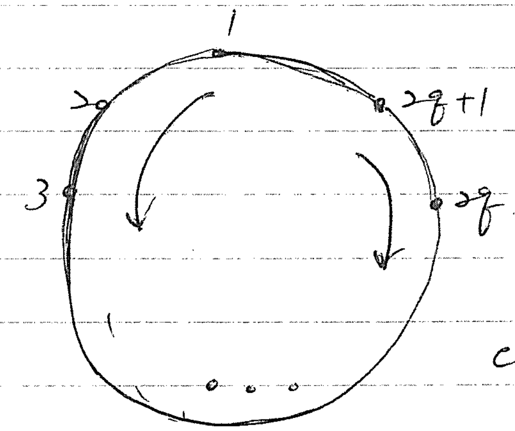


Theorem 97

Let G be a graph of size q and G has a graceful labeling. Then, K_{2q+1} can be decomposed into $2q+1$ copies of G.

Proof. Let $V(K_{2q+1}) = \mathbb{Z}_{2q+1}$. By arranging the vertices on a cycle, see Figure below, we notice that any two vertices have a circular distance at most q. More precisely, dist(i,j) (for $j>i$) $= \min\{j-i, (2q+1)-(j-i)\}$.

Now, we can add the labels of G for each one of them (taking modulo $2q+1$) and obtain the desired decomposition.

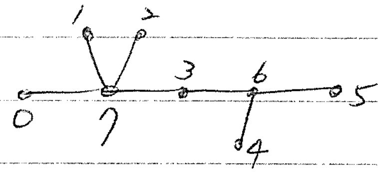


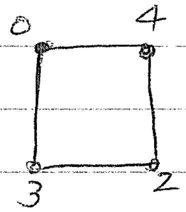
cicular distance

(*) If G has a graceful labeling f and the labeling has an extra property such that $\exists\ c_f \in \mathbb{R}$ satisfying for each uv either $f(u) \geq c_f > f(v)$ or $f(v) \geq c_f > f(u)$, then G has an α-labeling.

The idea of Theorem 97 is known as "difference method" and G with labeling is a "base graph".

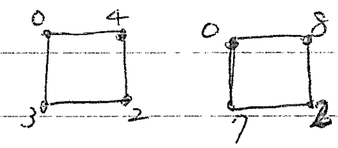
Example

The labeling of  is an α-labeling since we can choose $c_f = 3$ or 4 or 3.5. The following labeling is also an α-labeling; $c_f = 2.5$.



$\Rightarrow K_9 \xrightarrow{2} C_4$'s

$\Rightarrow K_{17} \xrightarrow{2} C_4$'s



Theorem 98

If G has an α-labeling and $\|G\| = q$, then $G \mid K_{2tq+1}$ where $t \in \mathbb{N}$. ($G \mid K_{2tq+1}$ denotes K_{2tq+1} has a G-decomposition.)

Proof. By Theorem 97, $G \mid K_{2q+1}$ can be obtained by a graceful labeling of G. Now, if G has an α-labeling, we may change the labels to find t base graphs for the decomposition of K_{2tq+1}. As mentioned above on the case of C_4's, for each label larger than c_f, we add $q, 2q, \cdots, (t-1)q$ respectively. This gives a collection of t base graphs (with labels). By difference method, we have the proof. (All differences from 1 to tq have been used exactly once.) ■

(*) If G has an α-labeling, then G must be a bipartite graph. The two partite sets of G are obtained by using the labels, larger than q and smaller than q respectively.

(∘) A graph G may have β-labeling but not α-labeling.

(**) One of the most beautiful conjectures on labelings is

<u>Graceful Tree Conjecture</u>: Every tree has a graceful labeling.

Of course, you may also conjecture that every tree has an α-labeling (but this is in general not true).

(∘) We have shown that for each graph G there exists a $\Delta(G)$-regular graph H such that $G \preceq H$. In fact, we can say more about this type of supergraph.

<u>Theorem 99</u>

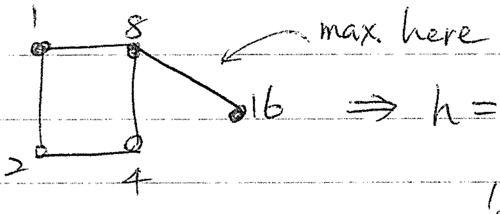
Let G be a graph of size q without isolated vertices. Then, there exists a regular graph H such that $G \mid H$. More precisely, H is a $2q$-regular graph.

Proof. Let $G = \{v_1, v_2, \cdots, v_p\}$ and f is a labeling of G such that $f(v_i) = 2^{i-1}$. Then, all edges will receive distinct weights $|f(u) - f(v)|$. We shall construct a $2q$-regular graph of order h such that

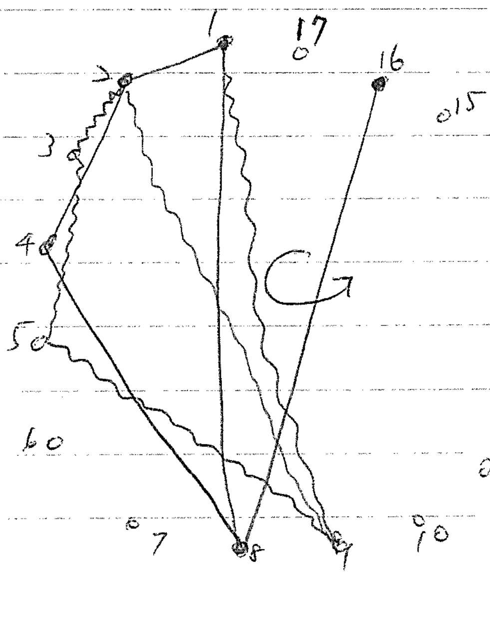
$$h = 1 + 2\cdot\max\{\,|f(v_i) - f(v_j)| \mid v_iv_j \in E(G)\}.$$

Let $V(H) = \mathbb{Z}_h$ and G be the graph with its vertices the labels from f, see the following figure for an example. Then, by difference method, the base graph will generate a regular graph which use the weights in G exactly once. Since $\|G\| = q$, the graph obtained H will be a $2q$-regular graph of order h. ∎

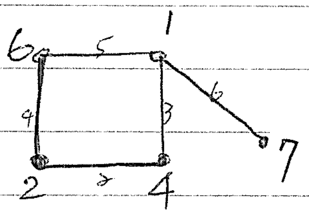
(*) We can decrease the order of H by giving another labeling satisfying all $|f(u) - f(v)|$'s are different for $uv \in E(G)$.

G: 

$\Rightarrow h = 17$



Another example



H : 10-regular graph of order 13.

(•) "Graph Decomposition" is one of the most important topic in Graph Theory, many results are also related to the study of Combinatorial Designs.

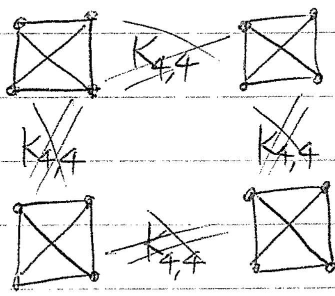
(*) Problem For which graph G, $K_3 | G$?

(*) Problem For which graph G, $C_4 | G$?

Theorem There exists a graph G with $\delta(G) < \frac{3}{4}|G|$ such that $K_3 \nmid G$.

Proof. For general n, the construction is similar to the following graph of order 16.

G :



$\Delta(G) = 11 < 12$

Since their are four bipartite subgraphs $K_{n,n}$ in G, the K_3-decomposition needs to use up all these edges by using one from K_n and two from $K_{n,n}$. ($K_{n,n}$ contains no odd cycles!) Hence, we need at least $\frac{1}{2}(4n^2)$ edges from four K_n's. But, $4K_n$ has $2n(n-1)$ edges which are not enough! The K_3-decomposition of G is not possible. $\blacksquare$

<u>Nash-Williams Conjecture</u>

For any graph G of order p, G has a K_3-decomposition provided $\delta(G) \geq \frac{3}{4}p$ and $3 \mid \|G\|$.

How about C_4-decomposition? Keep moving forward!