Lecture 19 (For reference) ②

Theorem 91

Let G be a (p,q)-graph and $A(G) = A$. Then,

(1) the number of triangles in G is $\frac{1}{6}\, tr(A^3)$;

(2) the number of 4-cycles in G is $\frac{1}{8}\left[tr(A^4) - 2q - \sum_{i \neq j} a_{i,j}^{(2)}\right]$

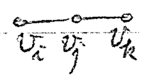
where $a_{i,j}^{(2)}$ is the (i,j)-entry in A^2; and

(3) the number of 5-cycles in G is $\frac{1}{10}\left[tr(A^5) - 5\,tr(A^3) - \right.$

$\left. 5 \sum_{i=1}^{p} \sum_{j=1}^{p} (a_{i,j} - 2) \cdot a_{i,i}^{(3)}\right]$.

Proof. It follows from the fact that the number of walks of length k from v_i to v_j is equal to $A^{(k)}(i,j)$. This can be proved by induction on k.

Hence, if triangles are concerned, then we consider $A^{(3)}(i,i)$, i.e., $tr(A^3)$. Since for each triangle (v_i, v_j, v_k), there are 6 different ways of 3-walks: $\langle v_i, v_j, v_k, v_i\rangle$, $\langle v_i, v_k, v_j, v_i\rangle$, $\langle v_j, v_k, v_i, v_j\rangle$, $\langle v_j, v_i, v_k, v_j\rangle$, $\langle v_k, v_i, v_j, v_k\rangle$ and $\langle v_k, v_j, v_i, v_k\rangle$, the result follows by using $\frac{1}{6}\, tr(A^3)$.

For 4-cycles and 5-cycles, we have to take away those 4-walks (and 5-walks) which are not for cycles, for example 

$\Rightarrow \langle v_i, v_j, v_k, v_j, v_i\rangle$.

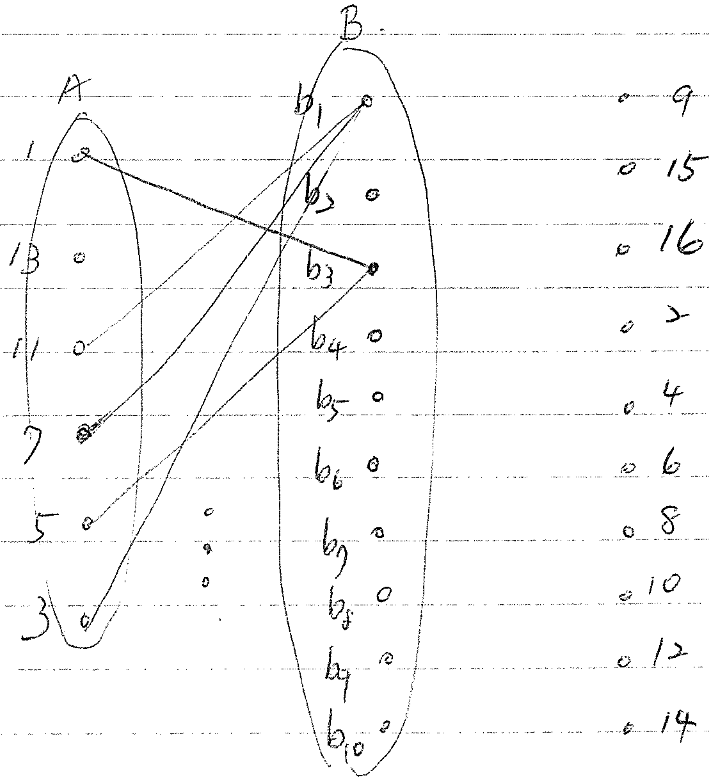
Check (2) and (3) yourself. ▨

Theorem 92

Let $G=(A,B)$ be a tree of order at most 16. Then, G has a prime labeling.

Proof (Sketched). We present a proof of the case $|A|=6$ and $|B|=10$. First, we label A by using $S=\{1,13,11,7,5,3\}$. Then, the labeling of the vertices can be found from the set $\underbrace{\{1,2,\cdots,16\}}_{[16]}\setminus S$. Now, denote $B=\{b_1,b_2,\cdots,b_{10}\}$. We claim that there exists a labeling of the vertices of B by using $[16]\setminus S$.

Let $B_i=\{x\in[16]\setminus S \mid \gcd(x,s)=1 \text{ for each } x\in S\cap N_G(b_i)\}$. Now, consider $\underbrace{\left|\bigcup_{j=1}^{k} B_{i_j}\right|}_{|S_k|}$. Clearly, if $k=1$, then $|S_k|\geq 1$ since $\gcd(2,s)=1$ for each $s\in S$. In fact,



each B_i contains at least 4 elements, 2, 4, 8 and 16.

By the fact, any two B_i's have at most one common neighbor, we can verify that $|S_k|\geq k$ for $6\leq k\leq 10$. Hence, By Hall's condition, the proof follows. ∎

Theorem 93 (Alon, 1990)

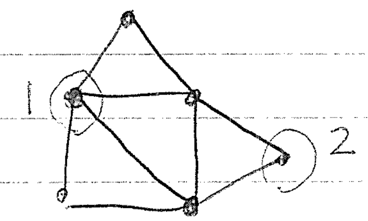
Let G be a graph of order p. Then, $\gamma(G)$ (the domination number) of G, $\gamma(G) \le n[1+\ln(\delta(G)+1)]/(\delta(G)+1)$.

Proof. (Probabilistic Method)

Let S be a subset of $V(G)$ with the probability of each vertex $p =_{def} \frac{\ln(\delta(G)+1)}{\delta(G)+1}$. Let $T = \{x \mid x \notin S, N_G(x) \cap S = \phi\}$. Since for each $y \notin S \cup T$, $N_G(y) \cap S \neq \phi$, $S \cup T$ is a dominating set of G. By the expectation of $E(S \cup T) = E(S) + E(T) \le np + n\cdot(1-p)^{\delta(G)+1}$ $\le np + n \cdot e^{-p(\delta(G)+1)} = n(p + \frac{1}{\delta(G)+1})$. This implies that there exists a dominating of size at most $n \cdot [1+\ln(\delta(G)+1)]/(\delta(G)+1)$. ■

(◦) A greedy algorithm for finding the dominating set:

Choose the vertices of a dominating set one by one following the idea: A vertex that covers the maximum number of vertices which are not cover yet is pick.



p is not a constant.

Theorem 94 (Omit the proof)

Let $p = p(n)$. Then, we have

(1) $p(n) = n^{-2}$ $\longrightarrow$ No edges.

(2) $p(n) = n^{-\frac{3}{2}}$ $\longrightarrow$ G has a nontrivial component which grows like a tree.

(3) $p(n) = n^{-1}$ $\longrightarrow$ Contain a cycle.

(4) $p(n) = \frac{\ln n}{n}$ $\longrightarrow$ Connected.

(5) $p(n) = (1+\varepsilon)\cdot\frac{\ln n}{n}$ $\longrightarrow$ Contains a Hamilton cycle.

(*) The growth rate is getting smaller.

Theorem 95 (More about eigenvalues of $A(G)$)

Let G be a connected graph of order p and A be its adjacency matrix. Then, we have the following basic properties.

(1) For each eigenvalue λ, $|\lambda| \le \Delta(G)$.

(2) $\Delta(G)$ is an eigenvalue of G if and only if G is regular.

Moreover, if $\Delta(G)$ is an eigenvalue of G, then the multiplicity of $\Delta(G)$ is 1.

(3) If $-\Delta(G)$ is an eigenvalue of G, then G is regular and bipartite.

(4) If G is bipartite and λ is an eigenvalue then $-\lambda$ is also an eigenvalue, moreove, they have the same multiplicity.

(5) The maximal eigenvalue $\lambda_{max}(G)$ satisfies $\delta(G) \le \lambda_{max}(G) \le \Delta(G)$.

(6) If $H \trianglelefteq G$, then $\lambda_{min}(G) \le \lambda_{min}(H) \le \lambda_{max}(H) \le \lambda_{max}(G)$.

Proof.

(1) Let $\vec{x}$ be an eigenvector with eigenvalue λ, i.e., $A\vec{x} = \lambda\vec{x}$.

Let $\vec{x} = (x_1, x_2, \cdots, x_p)$ and $|x_i| \le 1$ (by re-scaling $\vec{x}$). Suppose that $|x_j| \ge |x_i|$ for each $i = 1, 2, \cdots, p$. For convenience, let $x_j = 1$.
($|x_j|$最大，则令 $\vec{x} \leftarrow \vec{x}/\pm|x_j|$.)

Then, $|\lambda| = |\lambda \cdot x_j|$

$$= \left|\sum_{i=1}^{p} a_{j,i} \cdot x_i\right| \le \sum_{i=1}^{p} a_{j,i} |x_i| \le |x_j| \cdot \deg_G(v_j) \le \Delta(G). \blacksquare$$

(2) If $\Delta(G)$ is an eigenvalue, then as in (1), let $|x_j| = 1$, and we have $\Delta = \Delta \cdot x_j = \sum_{i=1}^{p} a_{j,i} x_i$. Hence $x_i = x_j = 1$ and $\deg_G(v_j) = \Delta$ whenever $v_i \sim_G v_j$. Therefore, $\deg_G(x_i) = \Delta$. Now, by the same argument, if $v_k \sim v_i$, $\deg_G(v_k) = \Delta$, then G is Δ-regular by

the fact that G is connected. This also implies that the eigenvector is $\vec{1} = (1, 1, \cdots, 1)$. The reverse statement is easy to see.

(3) If $-\Delta(G)$ is an eigenvalue, then by (2) we have $\deg_G(v_j) = \Delta$ and $x_i = -x_j = -1$ whenever $v_i \sim_G v_j$. Since two vertices are adjacent if they have distinct weights (x_i and x_j) 1 and -1, the vertex set of G can be partitioned into two subsets V_1 and V_2 such that $v_i \sim_G v_j$ iff their corresponding weights are different (1 or -1). Hence, G is bipartite.

(4) It follows by considering $Ker(A - \lambda I_p)$ and $Ker(A + \lambda I_p)$.

Let $G = (V_1, V_2)$ and $\vec{b} = (b_1, b_2, \cdots, b_p)$ such that $b_i = 1$ if $v_i \in V_1$ and $b_i = -1$ if $v_i \in V_2$. Now, if $A\vec{x} = \lambda \vec{x}$ and $v_i \in V_1$, then

$$A \cdot (\underbrace{\vec{b} \otimes \vec{x}}_{\substack{\overset{def}{\parallel} \\ (b_1 x_1,\, b_2 x_2,\, \cdots,\, b_p x_p)}})_i = \sum_{j=1}^{p} a_{ij} \cdot b_j x_j = \underbrace{\sum_{v_j \in V_1} a_{ij} x_j}_{\substack{\parallel \\ 0 \ (v_i \in V_1)}} - \sum_{v_j \in V_2} a_{ij} x_j$$

$$= -\sum_{v_j \in V_1} a_{ij} x_j - \sum_{v_j \in V_2} a_{ij} x_j = -\sum_{j=1}^{p} a_{ij} x_j = -\lambda x_i = -\lambda (\vec{b} \otimes \vec{x})_i$$

This implies λ and $-\lambda$ occur the same number of times in solving $A\vec{x} = \lambda \vec{x}$, i.e., $m(\lambda) = m(-\lambda)$.

$\downarrow$ multiplicity of λ

(5) By (1), we have $\lambda_{max}(G) \leq \Delta(G)$. Now, we claim the other inequality. Let the numerical range of A be $V(A)$, i.e., $V(A) = \{ \langle A\vec{x}, \vec{x} \rangle = \vec{x}^{t} A \vec{x} \mid |\vec{x}| = 1 \}$. Hence, let $\vec{1} = (1, 1, \cdots, 1)$ and we have $\frac{1}{p} \langle A\vec{1}, \vec{1} \rangle \in V(A)$.

Now, $\lambda_{max} = \max V(A) \geq \frac{1}{p} \langle A\vec{1}, \vec{1} \rangle = \frac{1}{p} \sum_{k=1}^{p} \deg_G(v_k) \geq \delta(G)$. ▨

(6) Let H be an induced subgraph of order $p-1$, i.e., $H = \langle \{v_1, v_2, \cdots, v_p\} \rangle_G$

Then, $\lambda_{max}(H) = \langle A'\vec{y}, \vec{y} \rangle$ where $A' = A(H)$ and $\langle \vec{y}, \vec{y} \rangle = 1$.

Now, consider $\vec{x} = (y_1, y_2, \cdots, y_{p-1}, 0)$ where $\vec{y} = (y_1, y_2, \cdots, y_{p-1})$.

Clearly, $\langle A\vec{x}, \vec{x} \rangle = \underbrace{\langle A'\vec{y}, \vec{y} \rangle}_{= \lambda_{max}(H)}$ and $\langle \vec{x}, \vec{x} \rangle = 1$. Since $\langle A\vec{x}, \vec{x} \rangle \in V(A)$, i.e., $\lambda_{max}(H) \in V(H)$. This implies that $\lambda_{max}(G) \geq \lambda_{max}(H)$. The other inequalities can be shown similarly. ▨