Theorem 86

For each bipartite graph G, there exists a graph $\tilde{G}$ such that $G \preceq \tilde{G}$ and $\tilde{G}$ is $\Delta(G)$-regular.

Proof. Let $G = (A, B)$ with $|A| \leq |B|$.

Let $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$. $(m \leq n)$

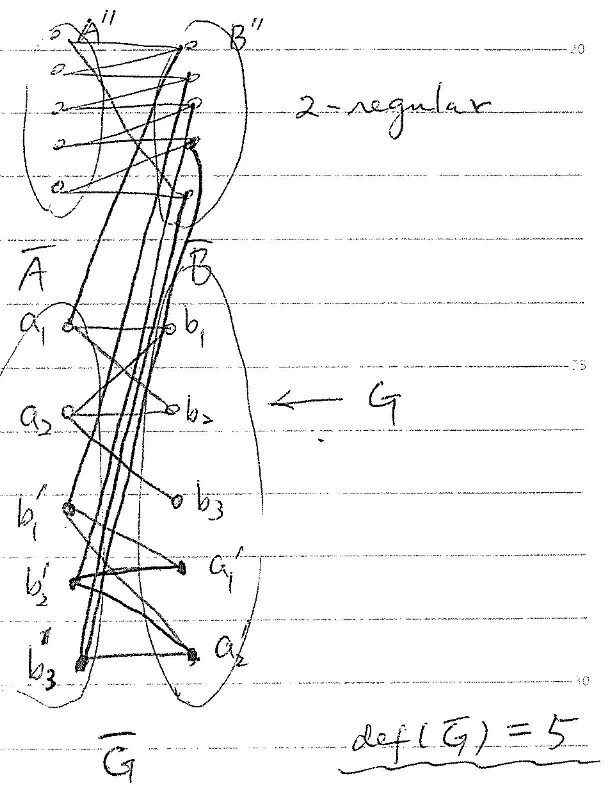
First, we construct a graph $\bar{G} = (\bar{A}, \bar{B})$: $(A \cup B', B \cup A')$ where

$A' = \{a_1', a_2', \cdots, a_m'\}$ and $B = \{b_1', b_2', \cdots, b_n'\}$ and $E(\bar{G}) = E(G) \cup \{\{b_j', a_i'\} \mid a_i b_j \in E(G)\}$. (See Figure for example.) In fact,

$\langle A \cup B \rangle_{\bar{G}} \cong \langle A' \cup B' \rangle_{\bar{G}}$. Hence, $\sum_{v \in A \cup B'} deg_{\bar{G}}(v) = \sum_{v \in B \cup A'} deg_{\bar{G}}(v) = 2\|G\|$,

$|\bar{A}| = |\bar{B}|$, $\sum_{v \in A \cup B'} (\Delta(G) - deg_{\bar{G}}(v)) = \sum_{v \in B \cup A'} (\Delta(G) - deg_{\bar{G}}(v)) = def(\bar{G})$.

Now, based on $def(\bar{G})$, we have two cases to consider.



__Case 1__ $def(\bar{G}) \geq \Delta(G)$

Construct a bipartite graph G'' such that $G'' = (A'', B'')$, $|A''| = |B''| = def(\bar{G})$ and G'' is $(\Delta(G) - 1)$-regular. The proof then follows by connecting A'' to $\bar{B}$ and B'' to $\bar{A}$.

<u>Case 2</u> $def(\bar{G}) < \Delta(G)$.

Let $|A''| = |B''| = \Delta(G)$, and $G'' \cong K_{\Delta(G),\Delta(G)} - M$, M is a matching of size $def(\bar{G})$. The proof follows by connecting the vertices in G'' (which are incident to M) and the vertices in $\bar{G}$ whose degrees are less than $\Delta(G)$.

So, the graph $\tilde{G}$ is defined on $(A'' \cup \bar{A}, B'' \cup \bar{B})$.

Moreover, $\tilde{G}$ is $\Delta(G)$-regular with partite set size $\max\{|A|+|B|+def(G), |A|+|B|+\Delta(G)\}$

<u>Theorem 87</u>

Let G be a planar graph with $\Delta(G) \geq 10$, then G is of Class 1.

<u>Proof.</u> We shall apply a lemma obtained by Vizing.

<u>Vizing's adjacency lemma</u>

<u>First form</u>: If G is of Class 2, then every vertex of G is adjacent to at least two major vertices. In particular, G contains at least three major vertices.

<u>Second Form</u>: Let G be a connected graph of Class 2 that is minimal with respect to edge coloring. <u>If $uv \in E(G)$ and $\deg_G(u) = m$, then v is adjacent to at least $\Delta(G) - m + 1$ major vertices.</u> (*)

We shall apply the 2nd form to prove the theorem.

Suppose not. Let G be a counterexample with minimum size. Thus, G is planar, $\overset{k}{\overset{\|}{\Delta(G)}} \geq 10$ and $\chi'(G) = k+1$. Clearly, G is minimal with respect to chromatic index. Since G is planar, G contains vertices of degree 5 or less, let S be the set of all such vertices. Define $H = G - S$. Again, H is planar, H contains a vertex w such that $\deg_H(w) \leq 5$. By the fact $\deg_G(w) > 5$, w is adjacent to some vertices of S. Let $vw \in E(G)$ where $v \in S$. In G, $\deg_G(v) \leq 5$. By (*), w is adjacent to at least $\Delta(G) - 5 + 1$ (≥ 6) vertices (major) of degree $\Delta(G)$. This implies that w is adjacent to at least 6 vertices of H since all major vertices are in H. Hence, $\deg_H(w) \geq 6$. $\rightarrow\leftarrow$

This concludes the proof.

(∘) $\sigma(G)$: Vertex cover number ; $\alpha_1(G)$: matching number

Theorem 88 (König–Egeváry) For bipartite graphs G, $\sigma(G) = \alpha_1(G)$.

Proof. We shall apply max-flow min-cut theorem to prove the theorem. First, we define a network as in Figure 1. Let $A = \{a_1, a_2, \cdots, a_m\}$, $B = \{b_1, b_2, \cdots, b_n\}$ and $G = (A, B)$. Then, the network is defined by letting u and v be source and sink respectively, $cap(u, a_i) = 1$ for $i = 1, 2, \cdots, m$, $cap(b_j, v) = 1$ for $j = 1, 2, \cdots, n$, and $cap(a_i, b_j) = |G| + 1$ if $a_i b_j$ is in $E(G)$. (Note that all arcs are from A to B.

Since $\sigma(G) \geq \alpha_1(G)$ as mentioned above, it suffices to show that $\alpha_1(G) \geq \sigma(G)$.

Now, let f be a maximum flow, It is easy to see that $\text{val} f = \alpha_1(G)$. This is due to the fact that all the arcs from u and into v are of capacity 1. (No two arcs can be out of two vertices in A and ended in a vertex of B.)

So, it is left to consider the minimum cut, let it be $K = (X, \overline{X})$ where $u \in X$, $v \in \overline{X}$, $A \cap \overline{X} = A'$ and $B \cap X = B'$, see Figure 2. Hence, K contains arcs from u to A', $A \setminus A'$ to $B \setminus B'$ and B' to v. Notice that $cap K \leq |G|$, for example, let $X = \{u\}$.

This implies that the following (⊛) is true and $A' \cup B'$ is a vertex cover and $\text{cap}\, K = |A'| + |B'| = \text{val}\, f = \alpha_1(G)$. By the fact, $|A'| + |B'| \geq \alpha(G)$, we have $\alpha_1(G) \geq \alpha(G)$ ■

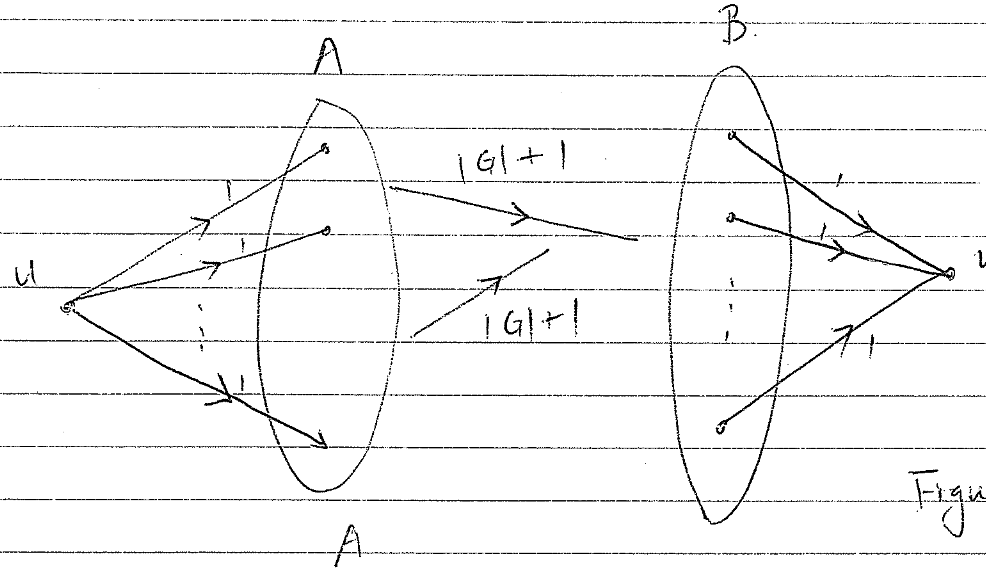


Figure 1. Network

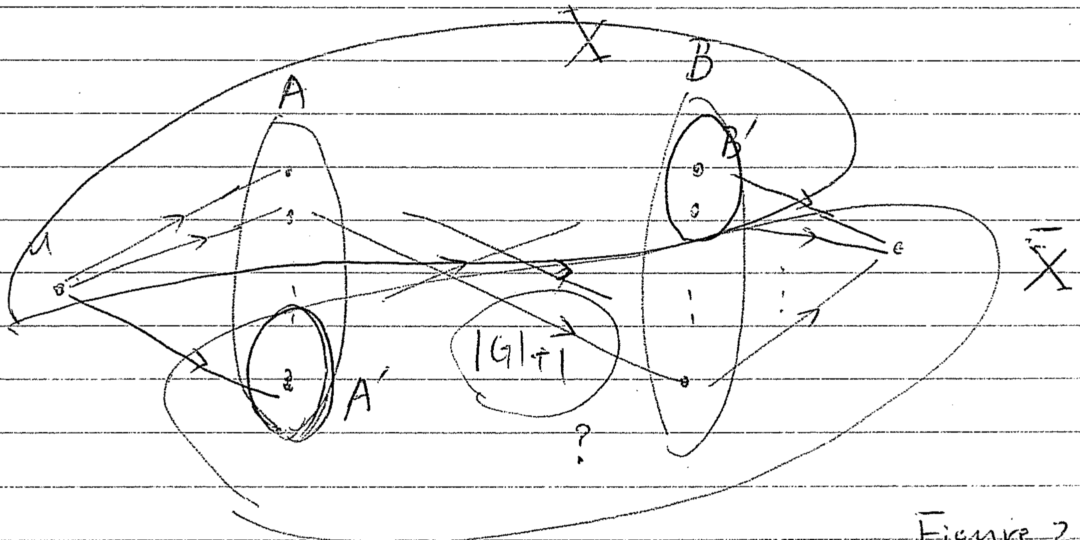


Figure 2. Cut

(⊛) In $(X, \bar{X})$, there exist no edges from $A \setminus A'$ to $B \setminus B'$. For otherwise, it is not a minimum cut. This implies that all edges are incident vertices in $A' \cup B'$. $A' \cup B'$ is a vertex cover.

<u>Theorem 89</u>

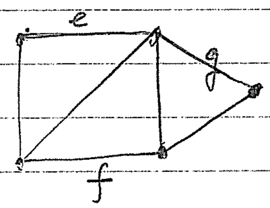
Let G be a graph of order p which has no isolated vertices.
Then, $\alpha(G) + \sigma(G) = p$.

<u>Proof.</u> Let S be a vertex cover (with $\sigma(G)$ vertices) of G. Then, $V(G) \setminus S$ is an independent set. Hence, $|V(G) \setminus S| \leq \alpha(G)$ and thus $p - |S| \leq \alpha(G)$. This implies that $p \leq \alpha(G) + |S| = \alpha(G) + \sigma(G)$. On the other hand, let T be an independent set of G such that $|T| = \alpha(G)$. Then, $G - T$ is vertex cover of G. By the fact $\underset{(\text{min.})}{\sigma(G)} \leq |G - T| = p - \alpha(G)$, we have $p \geq \sigma(G) + \alpha(G)$. ∎

<u>Definition</u> (Edge-cover)

An edge cover of a graph is a set of edges M such that all vertices of G are incident to M, i.e, for each $v \in V(G)$, v is incident to an edge in M.

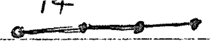
e.g. $\{e, f, g\}$ is an edge cover.



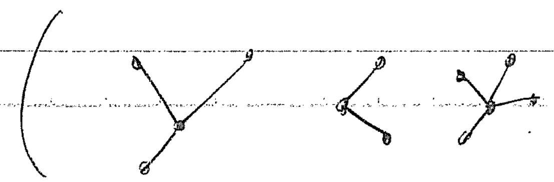
The edge cover number of G, $\sigma_1(G) = \underline{\min}\{|M| \mid M \text{ is an edge cover}\}$

(∘) $\sigma_1(G) \geq \lceil \frac{|G|}{2} \rceil$. (Each edge can cover two vertices.)

Theorem 90 $\sigma_1(G) + \alpha_1(G) = p$. ($G$ is a connected graph.)

Proof. Let M be a matching in G with $\alpha_1(G)$ edges. Then, for each vertex not in M, v, is incident to a vertex in M if v is in an edge of G. Assume that there t vertices not in M, i.e., $p = 2|M| + t$. Now, by taking every edge in the matching M and the set of t edges not in M but incident to M, we have an edge cover with $|M| + t$ edges. This implies that $\sigma_1(G) \leq |M| + t$. As a consequence, we have $p = 2|M| + t = \alpha_1(G) + |M| + t \geq \alpha_1(G) + \sigma_1(G)$.
On the other hand, let N be an edge cover of G with minimum number of edges, i.e, $|N| = \sigma_1(G)$. Notice that $\langle N \rangle_G$ is a disjoint union of stars. (You can not find P_4  in $\langle N \rangle_G$.) Assume that there are t stars. Then, $p = |N| + t$. By the fact that in $\langle N \rangle_G$ we can find a matching of size t, $p \leq |N| + \alpha_1(G)$. ($\alpha_1(G) \geq t$.)

↑ maximum matching number

( $|N| = 9$, $t = 3$, $p = 12$)

Theorem 90′

Let G be a graph with $V(G)=\{v_1, v_2, \cdots, v_p\}$. Then, we can use the vertices of G as variables to obtain a generating function for all dominating sets of G.

Proof. Let f be defined as follows;

$$f(v_1, v_2, \cdots, v_p) = \prod_{i=1}^{p} \left(v_i + \sum_{u \in N_G(v_i)} u\right).$$

Then, each summand is a product $v_1^{c_1} v_2^{c_2} \cdots v_p^{c_p}$ where $0 \le c_j \le p$. Now, let $S = \{v_j \mid c_j > 0,\ j=1,2,\cdots,p\}$. If $u \in V(G) \setminus S$, say $u = v_k$, then in the product $v_1^{c_1} v_2^{c_2} \cdots v_p^{c_p}$, $c_k = 0$. But, one of its neighbor has been selected. This implies that v_k is incident to a vertex of S. ∎

(∘) For small order graphs, this is a good way to find dominating sets. In fact, the term with maximum of "0" in powers provide a dominating set with minimum size and thus the domination number is determined.