Lecture 17

Following the arguments from Theorem 80, we are able to show, if n is large enough, then a random graph obtained from constant p will give a graph with high connectedness.

<u>Theorem 81.</u> For every constant $p \in (0,1)$ and $k \in \mathbb{N}$, almost all graphs are k-connected.

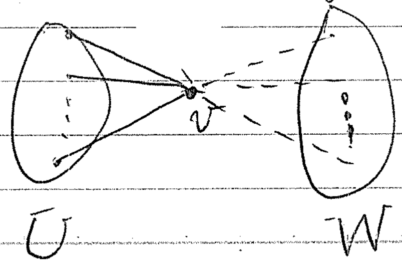
<u>Proof.</u> <u>First, we claim G^p has property "$P_{i,j}$".</u> ($P_{i,j}$ is the property that for any disjoint vertex sets U and W with $|U| \le i$ and $|W| \le j$, there exists a vertex $v \notin U \cup W$ such that $U = N_G(v)$ but $W \cap N_G(v) = \emptyset$. (See Figure below) Let $q = (1-p)$. Then, the probability of such v is $p^{|U|} \cdot q^{|W|} \ge p^i \cdot q^j$. ( )

Hence, the probability of no such v exists is

$$(1 - p^{|U|} \cdot q^{|W|})^{n-|U|-|W|} \le (1 - p^i \cdot q^j)^{n-i-j} \quad \text{for } n \ge i+j.$$

Now, there are at most n^{i+j} $\langle U, W \rangle$ pairs, and thus the probability of "$\sim P_{i,j}$" is at most $n^{i+j} \cdot (1-p^i q^j)^{n-i-j}$. By the fact i and j are constants, we have the probability "0" when $n \to \infty$.

not $P_{i,j}$



To prove the theorem, let $i=2$ and $j=k-1$. Since almost all graphs G have property $P_{2,k-1}$. Let W be an arbitrary set of $k-1$ vertices. Then, for any two vertices $\widetilde{x,y} \in V(G)\setminus W$, then either x is adjacent to y or x and y have a common neighbor. Therefore, W is not a vertex cut of size $k-1$. This implies that G is k-connected. ∎

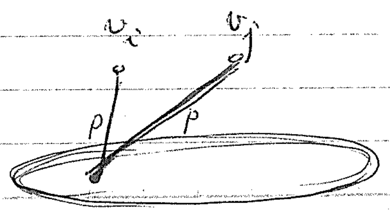
Not only the graph is with high connectivity, the graph does have very small diameter.

<u>Theorem 82</u> Almost all graphs are of diameter 2.

<u>Proof</u>. Let $X_{i,j}$ be the indicator random variable such that

$$X_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ do not have a common neighbor; and} \\ 0 & \text{otherwise.} \end{cases}$$

So, the random variable X of "no two vertices have a common neighbor is equal to $\sum_{i \neq j} X_{i,j}$. Hence, $\mathbb{E}(X) = \sum_{i\neq j} \mathbb{E}(X_{i,j}) = \binom{n}{2}\cdot(1-p^2)^{n-2} \to 0$. This implies that $\mathbb{E}(X) \Longrightarrow 0$ and we conclude that almost every pair of distinct vertices have a common neighbor. ∎



(*) We may replace p by $p(n)$. Then, we have a more complicat situation to consider the random graphs.

(**) We may also consider the probability of edges based on the vertices they are incident to. That is, the next edge will come from somewhere near a vertex with larger degree.

Problem Show that almost all graphs there is a unique vertex with maximum degree.

(○○○) If this is true, then almost all graphs are of Class 1, since Vizing did prove that a Class 2 graph contains at least three major vertices (degree $\Delta(G)$).

For convenience in referring this result, we put it as a theorem.

Theorem 83 For $p \in (0,1)$, almost all graphs obtained in Model A is of Class 1.

Proof.

Step 1. Prove that almost all graphs have a unique major vertex

Step 2. By Vizing's result, every Class 2 graph has at least three major vertices, we conclude the proof.

Some ideas in Algebraic Graph Theory.

(◦) The adjacency matrix $A(D)$ of a directed graph D is $A(D) = [x_{i,j}]_{n\times n}$ where $V(D) = \{v_1, v_2, \cdots, v_n\}$, $x_{i,j} = 1$ if and only if (v_i, v_j) is an arc of D.

(◦) If D is a graph (instead of digraph), then we view each edge of D as a pair of arcs in opposite directions, and thus $A(D)$ is a symmetric $(0,1)$-matrix.

(◦) If G is a simple graph, then we can define the <u>Laplacian</u> of G, denoted by $L(G) = [l_{i,j}]_{n\times n}$ where $l_{i,i} = \deg_G(v_i)$ and $l_{i,j} = -1 = l_{j,i}$ if $\{v_i, v_j\} \in E(G)$.

(◦) We shall consider $A(G)$ where G is a simple graph in what follow.

(◦) The characteristic polynomial of $A(G)$ is defined as

$$\phi(A, x) =_{def} \phi(G, x) = \det(xI_n - A). \quad (A \doteq A(G)).$$

(◦) The spectrum of A (matrix) is a list of its eigenvalues, the zeros of $\phi(A,x)$, together with their multiplicities.

<u>Example</u> $G \cong C_4$.

$\phi(C_4, x) = x^4 - 4x^2$; zeros are $2, 0, 0, -2$ (0 is of multiplicity 2).

(∘) The <u>largest</u> eigenvalue of a graph G is called the index of G and the spectral radius of G is the maximum value of $\{|\lambda_i| \mid i = 1, 2, \cdots, n, \text{ and } \lambda_i \text{ is an eigenvalue of } A\}$. In C_4 case, the index and spectral radius of G are equal.

(∘) $\mathrm{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_t \\ m_1 & m_2 & \cdots & m_t \end{pmatrix}$ and $\sum_{i=1}^{t} m_i = n$.

<u>Examples</u>

1. $\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$. $\qquad$ $\mathrm{diam}(K_n) = 1$

2. $\mathrm{Spec}(K_{m,n}) = \begin{pmatrix} \sqrt{mn} & 0 & -\sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}$. $\qquad$ $\mathrm{diam}(K_{m,2}) = 2$

<u>Theorem 84</u>

The diameter of a connected graph G is less than the number of distinct eigenvalues.

<u>Proof.</u> Let r be the number of distinct eigenvalues, let them be $\lambda_1, \lambda_2, \cdots, \lambda_r$. Then $\prod_{i=1}^{r}(x - \lambda_i)$ is the minimal polynomial

of A. This implies that the linear combination of $I_n = A^0, A^1, \cdots, A^r$ is the zero matrix. Now, consider $\mathrm{diam}(G)$. Let $d(v_i, v_j) = \mathrm{diam}(G) = k$. Then, $A^h(i,j) = 0$ for each $0 \le h < \mathrm{diam}(G)$. Hence, $A^0, A^1, \cdots, A^k$ are linearly independent. (?) This implies that $k < r$, i.e., $\mathrm{diam}(G)$ is less than the number of distinct eigenvalues. $\blacksquare$

<u>Theorem 85</u> For every graph G, $\chi(G) \le 1 + \lambda_{max}(G)$.

Proof. Let $\chi(G) = k$ and H be a vertex critical subgraph of G. That is, $\chi(H) = k$ and for each vertex $v \in V(H)$, $\chi(H - v) = k - 1$. By Theorem 56, $\delta(H) \ge k - 1$.

Now, consider λ_{max}. If λ is an eigenvalue of $A(G)$, then there exists an eigenvector $\vec{x}$ such that $A\vec{x} = \lambda \vec{x}$. Let $x_j = \max_{i=1}^{n} \{ x_i \mid \vec{x} = (x_1, x_2, \cdots, x_n) \}$. Then

$$\lambda x_j = (A\vec{x})_j = \sum_{v_i \in N(v_j)} x_i \le \deg_G(v_j) \cdot x_j \le \Delta(G) \cdot x_j .$$

Hence, $\lambda_{max} \le \Delta(G)$ and Theorem 85 is an improvement of Brooks Thm.

On the other hand, $\lambda_{max}(G) \ge \delta(G)$. (?) This implies that

$k \le 1 + \delta(H) \le 1 + \underline{\lambda_{max}(H)} \le 1 + \underline{\lambda_{max}(G)}$. $\blacksquare$

$$A(G) : \begin{bmatrix} \boxed{A(H)} & \\ & \end{bmatrix}$$

This course will stop here. Hopefully, the learning experience can provide you the basic knowledge of Graph Theory. For sure, you have learned the skills of "Research" through working on exercises. Keep Moving Forward!

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(∘) For each bipartite graph G, there exists a graph $\tilde{G}$ such that $G \preceq \tilde{G}$ and $\tilde{G}$ is $\Delta(G)$-regular.

Proof. Let $G = (A, B)$ with $|A| \le |B|$.

Let $A = \{a_1, a_2, \cdots, a_m\}$ and $B = \{b_1, b_2, \cdots, b_n\}$. $(m \le n)$

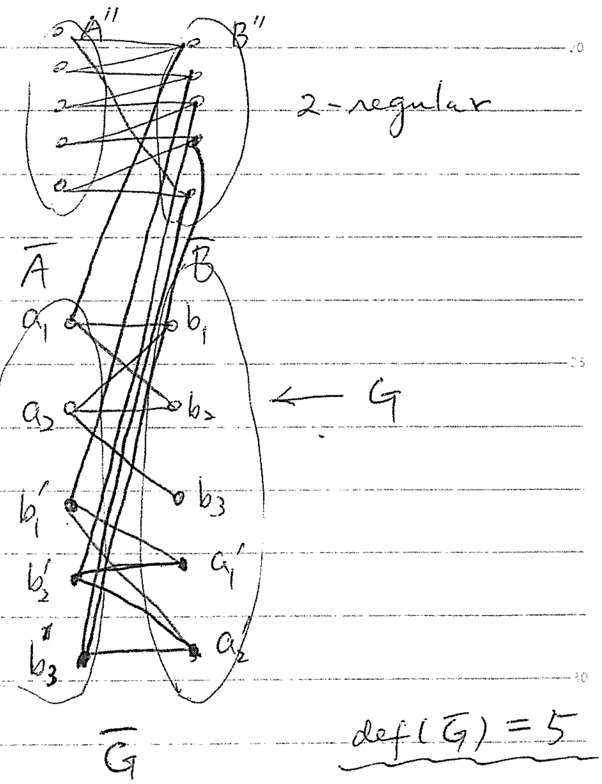
First, we construct a graph $\overline{G}$: $(A \cup B', B \cup A') = (\overline{A}, \overline{B})$ where

$A' = \{a_1', a_2', \cdots, a_m'\}$ and $B = \{b_1', b_2', \cdots, b_n'\}$ and $E(\overline{G}) = E(G) \cup \{\{b_j', a_i'\} \mid a_i b_j \in E(G)\}$. (See Figure for example) In fact,

$\langle A \cup B \rangle_{\overline{G}} \cong \langle A' \cup B' \rangle_{\overline{G}}$. Hence, $\sum_{v \in A \cup B'} \deg_{\overline{G}}(v) = \sum_{v \in B \cup A'} \deg_{\overline{G}}(v) = 2\|G\|$,

$|\overline{A}| = |\overline{B}|$, $\sum_{v \in A \cup B'} (\Delta(G) - \deg_{\overline{G}}(v)) = \sum_{v \in B \cup A'} (\Delta(G) - \deg_{\overline{G}}(v)) = \operatorname{def}(\overline{G})$.

Now, based on $\operatorname{def}(\overline{G})$, we have two cases to consider.



Case 1 $\operatorname{def}(\overline{G}) \ge \Delta(G)$

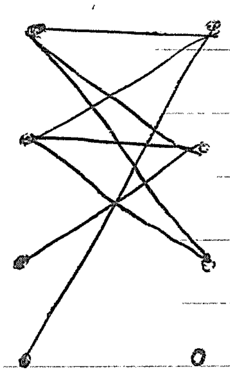
Construct a bipartite graph G'' such that $G'' = (A'', B'')$, $|A''| = |B''| = \operatorname{def}(\overline{G})$ and G'' is $(\Delta(G) - 1)$-regular. The proof then follows by connecting A'' to $\overline{B}$ and B'' to $\overline{A}$.

<u>Case 2</u> $def(\bar{G}) < \Delta(G)$.

Let $|A''| = |B''| = \Delta(G)$, and $G'' \cong K_{\Delta(G),\Delta(G)} - M$, M is a matching of size $def(\bar{G})$. The proof follows by connecting the vertices in G'' (which are incident to M) and the vertices in $\bar{G}$ whose degrees are less than $\Delta(G)$.

So, the graph $\tilde{G}$ is defined on $(A'' \cup \bar{A}, B'' \cup \bar{B})$.

Moreover, $\tilde{G}$ is $\Delta(G)$-regular with partite set size $\max\{|A|+|B|+def(G), |A|+|B|+\Delta(G)$



(?)