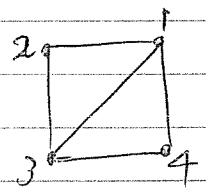
Lecture 16 Dec. 28 —

(∘) Prime Labeling

A prime labeling of a graph G is a mapping $f: V(G) \xrightarrow{1-1} \{1, 2, \cdots\}$ such that if $uv \in E(G)$, then $f(u)$ and $f(v)$ are relatively prime, i.e., $\gcd(f(u), f(v)) = 1$.

Example



(∘) $P(a,b)$: The set of primes between integers a and b.

$P(a,b]$, $P[a,b]$ and $P[a,b)$ can be defined accordingly.

(*) Bertrand's Postulate : (Bertrand–Chebyshev Theorem)

For each $n \geq 2$, $P(n, 2n) \neq \phi$.

(∘) $\alpha(G)$: Independence number of G.

$\sigma(G)$: Vertex cover number of G

(S is a vertex cover of G if every edge e of $E(G)$ is incident at least one vertex of S.) $\Rightarrow V(G) \setminus S$ is an independent set.

Theorem 26

Let G be a graph of order n. Then the followings hold.

(1) If $\alpha(G) < \lfloor \frac{n}{2} \rfloor$, then G has no prime labelings.

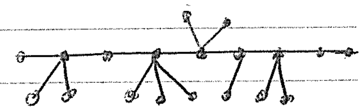
(2) If S is a vertex cover of G and $|S| \le |P(n, 2n)| + 1$, then G has a prime labeling.

Proof. (1) follows from the fact that in all prime labelings (if exist), the set of vertices with even labels induces an independent set. Hence, the graph must contain an independent set of size $\lfloor \frac{n}{2} \rfloor$.

(2) Use the labels 1 and $x \in P(n, 2n)$ to label the vertices in S, we obtain a prime labeling of G. (The vertices in $G - S$ can be labeled arbitrarily. ∎

Use the facts above, we can easily find a prime labeling of a tree of order at most 10. The proof follows by letting $T = (A, B)$ and consider the cases $|A| = 1, 2, 3, 4, 5$.

Prime labeling Conjecture



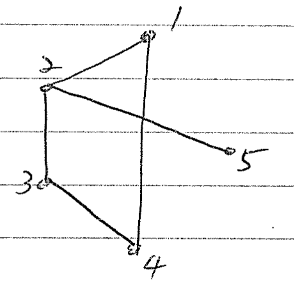
Every tree has a prime labeling.

(*) The first goal should be the answer for caterpillar.

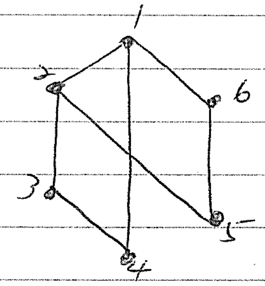
Prime sum graph

(i) The prime sum graph of order n, G_n, is defined on $[n]$ and two vertices i and j in $[n]$ are adjacent if $i+j$ is a prime.

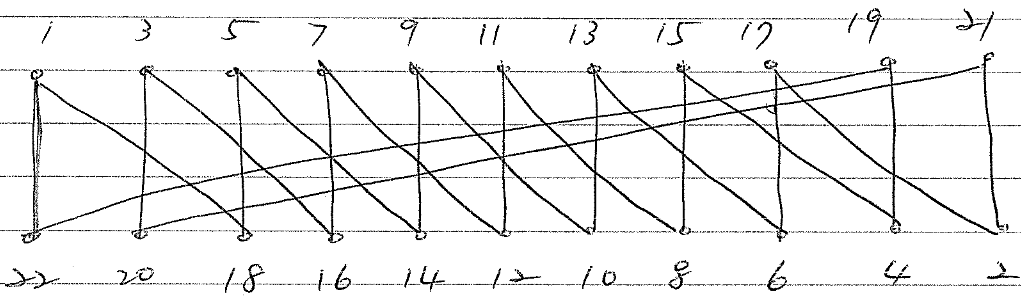
G_5:



G_6:



G_{22}:



A part of edges, 23, 19, 41 are primes used for them.

$(1, 22, 19, 4, 15, 8, 11, 12, 7, 16, 3, 20, 21, 2, 17, 6, 13, 10, 9, 14, 5, 18)$ ← Hamilton cycle.

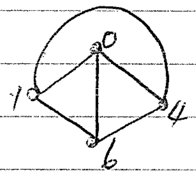
(ii) So, we are interested in determining "for which n, G_n has a Hamilton cycle?". Clearly n must be even. (?)

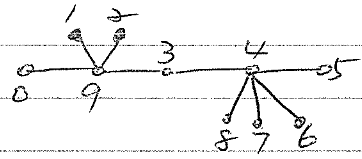
(*) It is not difficult to check, for small $n \geq 6$, G_n is indeed hamiltonia

(c) Graceful Labelings

(with $|G| \leq \|G\|+1$)

A graceful labeling of a graph G is a mapping $f: V(G) \xrightarrow{1-1} \{0,1,2,\cdots,\|G\|\}$ such that the weights of edges uv, defined by $|f(u)-f(v)|$, are all distinct. In case that $|G| > \|G\|+1$, we use the mapping $f: V(G) \xrightarrow{1-1} \{0,1,2,\cdots,|G|\}$ instead.

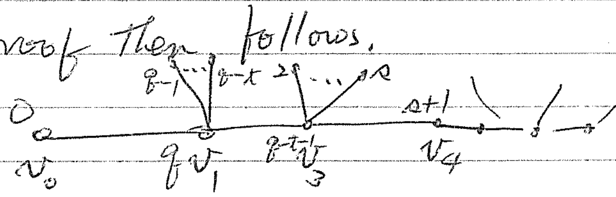




Theorem 22

Any caterpillar has a graceful labeling.

Proof. Starting from the end vertex from one side, label the vertex v_0 with 0, then the vertex v_1 incident to v_0 is labeled with $\|G\| = q$. Now, the neighbors of v_1 are labeled by $2, 3, \cdots$ for pendant vertices and let the largest label be used in v_3. Again, we shall start from the use of $q-1, q-2, \cdots$ and let the smallest label for v_4 to use. The proof then follows. ▨



(*) There are special trees which have a graceful labeling. But, to prove that all trees do have graceful labelings remains unsolved.

<u>Graceful Tree Conjecture</u> (Ringel – Kotzig)

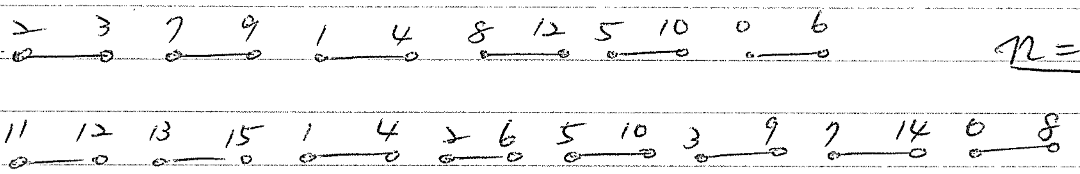
All trees are graceful.

(*) We remark here, for forests, we can also find graceful labelings.

<u>Theorem 11′</u>

Each matching of size n, M_n, has a graceful labeling.

<u>Proof.</u> This is a direct consequence of using Skolem sequences of order n. (See a couple of examples below.) ▨



$n=6$

$n=8$

(**) There are many problems in graph labelings, you may refer to the following reference for more informations.

A dynamic survey of graph labeling by J. A. Gallian.
(502 pages!)

Random Graphs

(*) The notion of random graphs is different from probabilistic method. Here are two theorems which uses probabilistic method.
(Find the lower bound of $R(s)$ is another example.

Theorem 28

There exists a tournament T_n such that T_n has at least $n!/2^{n-1}$ directed Hamilton paths.

Proof. In K_n, there are $n!$ Hamilton paths. (Starting from one of the vertices of K_n and choose one edge at a time without passing vertices which have been selected.) Now, if each edge is assigned with an orientation, the probability for any one Hamilton path to be a directed Hamilton path is $\frac{1}{2^{n-1}}$. (each edge has $\frac{1}{2}$ chance to be in the right direction.) So, if X is the random variable for the number of directed Hamilton paths, $E(X)$ (Expectation) $= n! \cdot \frac{1}{2^{n-1}}$ and therefore, there exists a tournament satisfying this value. ▨

Theorem 29

The independence number of G, $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{1+\deg_G(v)}$.

Proof. Let f be a random labeling of G by using $1, 2, \cdots, |G|$. For convenience, let $V(G) = \{v_1, v_2, \cdots, v_p\}$ and f is 1-1 mapping from $V(G)$ onto $\{1, 2, \cdots, p\}$. Now, for each v, there exists a unique $u \in N_G[v]$ such that $f(u) = \min\{f(x) \mid x \in N_G[v]\}$. Now, let S be the set of vertices v in $V(G)$ such that $f(v) = \min\{f(x) \mid x \in N_G[v]\}$. That's if v has the smallest label, then put v in S.

Now, clearly S is an independent set. (?) Moreover, the probability of being the smallest label among all its neighbors is $\frac{1}{\deg_G(v)+1}$, we conclude the proof since $\alpha(G) \geq |S| = \sum_{v \in V(G)} \frac{1}{1+\deg_G(v)}$. ∎

Random Graphs

There are different models, here we consider one of the most popular one.

Model A $G(n, p)$, $0 \leq p \leq 1$.

The probability of the existence of an edge (independently) is p and the graph induced by using existent edges is G_p.

(∘) We use G^n to denote the distribution of graphs of order n.

Let q_n be the probability of the existence of "property" Q when the graphs considered are of order n.

(∘∘) If $\lim_{n\to\infty} q_n = 1$, Then we say "Q" almost always holds. In this case, we say almost all graphs have property "Q".

<u>Theorem 80</u> (Gilbert, 1959)

Let p be a constant such that $0<p\le 1$. Then, almost all graphs are connected.

<u>Proof</u>. Suppose not. Then, there are graphs G (of order n) which is not connected. Hence, there exists a proper subset $S \subseteq V(G)$ such that $\langle S, V(G)\backslash S \rangle$ contains no edges. This implies that the probability q_n of the existence of disconnected graphs of order n satisfies

$$0 \le q_n \le \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} (1-p)^{k\cdot(n-k)} \cdot p^c \quad (c \text{ is a constant})$$

$$\le \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} n^k (1-p)^{k(n-k)} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(n\cdot(1-p)^{n-k}\right)^k$$

$$< \frac{y}{1-y} \quad (\text{where } y = (n\cdot(1-p)^{n-k})).$$

Since $\lim_{n\to\infty} n(1-p)^{n-k} = 0$, $\lim_{n\to\infty} \frac{y}{1-y} = 0$. Hence, $\lim_{n\to\infty} q_n = 0$ and thus almost all graphs are connected. ∎

(c) It is not difficult to see the connectedness should be quite strong, not only 1-connected. We can infact claim that any cut set of size $k-1$ is not available for a fixed k.