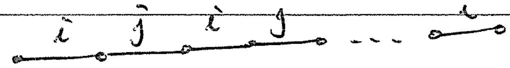
Graph Theory Lecture 15

Dec. 26 —

<u>Theorem 21</u> (Equitable edge-coloring)

If G has a k-edge-coloring f, then G has an equitable edge-coloring, i.e., for any two $i, j \in \{1, 2, \cdots, k\}$, $\big||f^{-1}(i)| - |f^{-1}(j)|\big| \le 1$.

<u>Proof.</u> If there exist i and j such that $|f^{-1}(i)| - |f^{-1}(j)| \ge 2$, then we consider the graph H induced by the set of edges colored i and j. Then, H is a subgraph of G such that each component of H is either a path or an even cycle. Since i occurs more times than j, there exists an i-j path :



whose end edges are colored i. Now, by switching the colors on this path, we obtain a new edge coloring of G such that i occurs one less time and j occurs one more. It turns out that we can obtain an k-edge-coloring s.t. $\big||f^{-1}(i)| - |f^{-1}(j)|\big| \le 1$. As a consequence, we are able to adjust all of them and obtain an equitable k-edge-coloring. ■

(*) This theorem is also not difficult to prove, but very useful.

(o) Without using 4CT, the proof of Theorem 70 is very difficult.

(*) It was conjecture that if G is planar and $\Delta(G)$ is large enough, say, ____.

Theorem 72

The complete graph K_n is of Class 2 if and only if K_n is overfull or equivalently n is odd.

Proof. First, we claim that for each $m \geq 1$, K_{2m} is of Class 1. It suffices to give a $(2m-1)$-edge-coloring of K_{2m}. For convenience, let $V(K_{2m}) = \mathbb{Z}_{2m} = \{0, 1, 2, \cdots, 2m-1\}$. For each color $i \in \{1, 2, \cdots, 2m-1\}$, let the set of edges colored i be

$$F_i = \{(0,i), (i+1, i-1), (i+2, i-2), \cdots, (i+m-1, i-m+1)\} \pmod{2m-1}.$$

See Figure 46 for an example of $m = 5$ and $i = 3$.

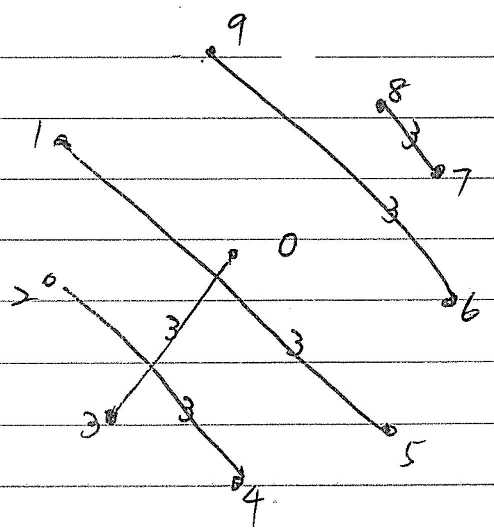


Figure 46. $\chi'(K_{10}) = 9$.

Since $\Delta(K_{2m}) = 2m-1$, $\chi'(K_{2m}) = 2m-1$.

Now, by deleting 0 in K_{2m}, we obtain a $(2m-1)$-edge-coloring of K_{2m-1}. On the other hand, it is not difficult to check that K_{2m-1} is overfull for $m \geq 2$, this concludes that $\chi'(K_{2m-1}) > \Delta(K_{2m-1}) = 2m-2$. ∎

(∘) This theorem is not difficult to prove, but it is very useful in the construction of "Combinatorial Designs".

(∘∘) Equivalently, K_{2m} can be decomposed into $2m-1$ 1-factors, which is also know as a 1-factorization of K_{2m}.

(∘∘∘) If G is an r-regular graph and $\chi'(G) = r$, then G has a 1-factorization.

(***) It was conjectured that if G is r-regular and $r \geq \frac{|G|}{2}$, then G has a 1-factorization or equivalently $\chi'(G) = r$.

<u>Theorem 21'</u> (D. Hoffman et al.)

A complete multipartite graph G is of Class 2 if and only if G is overfull.

(∘) Total coloring

A k-total coloring of a graph G is a mapping

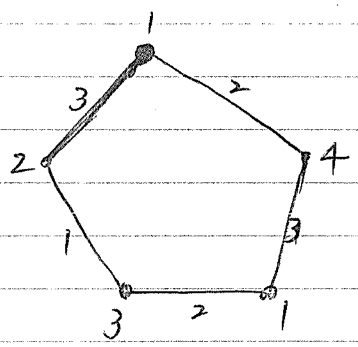
$\psi : V(G) \cup E(G) \longrightarrow \{1, 2, \cdots, k\}$ such that

(i) adjacent vertices receive distinct images,

(ii) incident edges receive distinct images, and

(iii) each vertex has a distinct image with the images of its incident edges.

e.g.

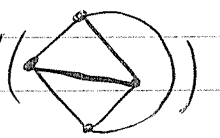


A 4-total coloring of C_5.

(∘) $\chi''(G) = \min.\{k \mid G$ has a k-total coloring$\}$.

(Total chromatic number)

Theorem 23

χ''() = 5 (?)

$\chi''(K_{2n+1}) = \chi''(K_{2n}) = 2n+1.$

Proof. $\chi''(K_{2n+1})$ can be obtained by using $\chi'(K_{2n+1}) = 2n+1$.

TCC Conjecture $\quad \chi''(G) \le \Delta(G)+2$.

Note that $\chi''(G) \ge \Delta(G)+1$. As to the total coloring of K_{2n}, we claim that $2n$ colors are not enough.

Observe that each color class has at most one vertex and $n-1$ edges. So, $2n$ color classes will contain at most $2n$ vertices and $2n(n-1)$ edges. Hence, there are $2n^2$ elements (vertices and edges) in total. But, K_{2n} has $2n + \frac{2n(2n-1)}{2}$ elements to color, which is $2n^2+n$. Clearly, $2n$ color is not enough. Since K_{2n+1} is $(2n+1)$-total colorable, K_{2n} is also $(2n+1)$-total colorable. The proof follows. ∎

(∘) Based on TCC Conjecture, a graph G is called Type 1 if $\chi''(G) = \Delta(G)+1$ and Type 2 otherwise.

Theorem 24. $K_{m,n}$ is of Type 1 if and only if $m \ne n$.

Proof. $(\Rightarrow)$ If $m=n$, then there are $2n+n^2$ elements to color. Since each color class contains at most n elements, $\Delta(G)+1 = n+1$ colors are not enough, $n(n+1) < 2n+n^2$. Hence, $\chi''(K_{n,n}) \ge n+2$, and thus $K_{n,n}$ is not of Type 1.

($\Leftarrow$)

On the other direction, let $m = n+k$. Now, $\Delta(K_{m,n}) = n+k$.
By the edge-coloring of $K_{m,n}$, we have an $n \times (n+k)$ Latin rectangle based on $\{1, 2, \cdots, n+k\}$, see Figure 47. Since $k \geq 1$, we may extend this rectangle to $(n+1) \times (n+k)$ and the last row can be used to color the vertices of A. Finally, color all vertices of B by one extra color, we have $\chi''(K_{m,n}) \leq n+k+1 = \Delta(K_{m,n})+1$. ∎

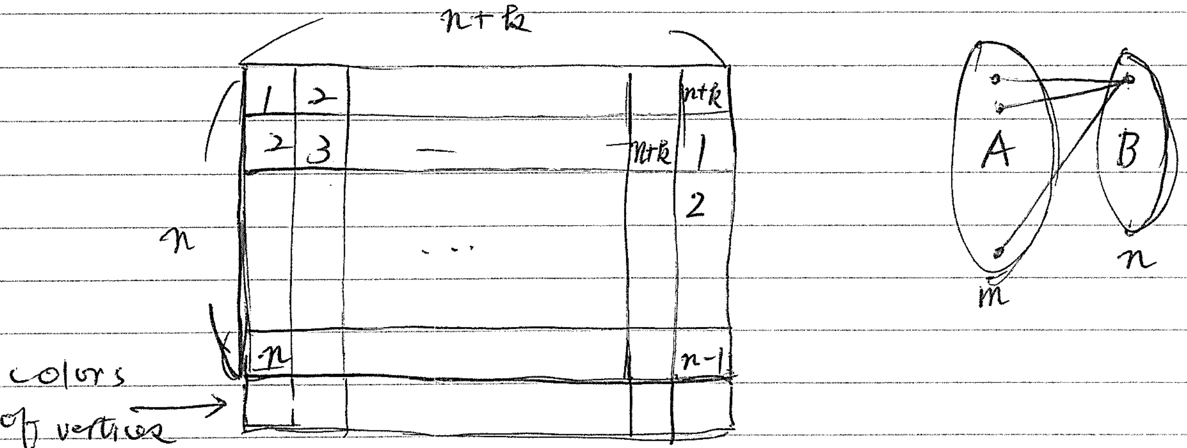
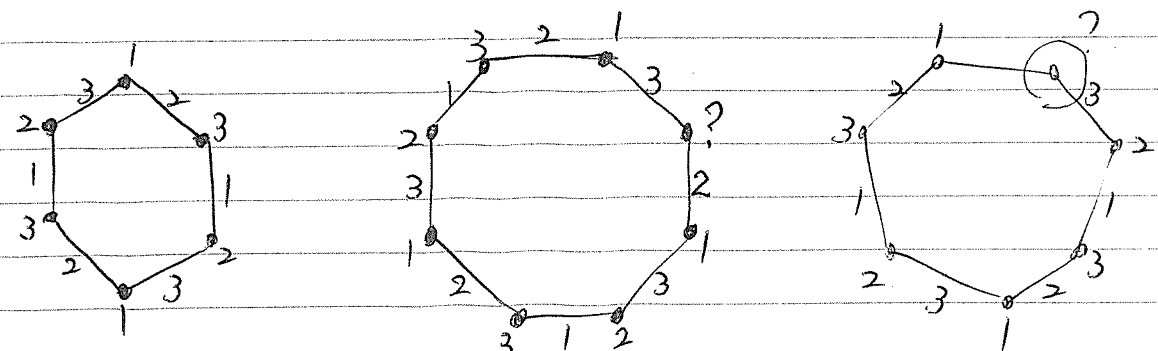


Figure 47, An edge-coloring of $K_{m,n}$.

(o) A cycle C_n is of Type 1 if and only if $n \equiv 0 \pmod 6$.



If we use three colors, then starting from one vertex and one edge, all the colors of the others are forced!

(∘) The deficiency of a graph G, $def(G)$, is defined as $\sum_{v \in V(G)} (\Delta(G) - \deg_G(v))$.

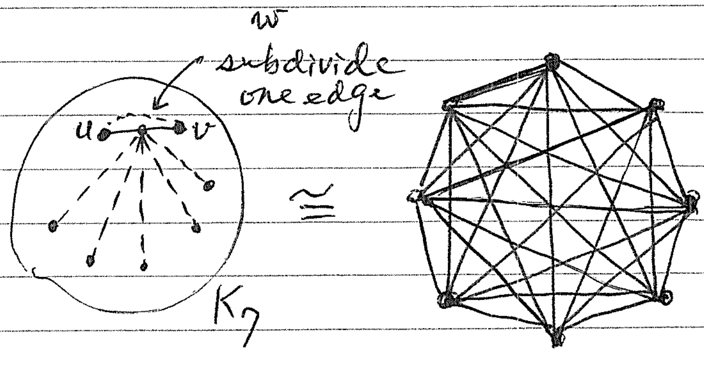
(∘∘) G is conformable if G has a vertex coloring $\varphi : V(G) \to \{1, 2, \dots, \Delta(G)+1\}$ such that $def(G) \geq |\{ i \mid |G| - |\varphi^{-1}(i)| \equiv 1 \pmod 2\}|$.

Theorem 75

Let S_i be a star with i edges. Then, $K_{2n} - S_1 - S_{2n-3}$ ($= G(1, 2n-3)$) is of Type 2.

Proof. Assume that $G(1, 2n-3)$ is of Type 1, i.e, there exists a total coloring ψ of $G(1, 2n-3)$ using $\Delta(G) + 1 = 2n-1$ colors. Let uv be the edge subdivided by w, see Figure 48. First, if $\psi(u) = \psi(v) \neq 0$, then let $\psi(w) = 1$, $\psi(uw) = 2$ and $\psi(wv) = 3$. Let r_j the number of vertices v in which j occurs in either v or an edge incident to v. Hence, $\sum_{j=0}^{2n-2} r_j = |K_{2n}| + 2\|G(1, 2n-3)\|$. (?) Now, 0 occurs in at most $2n-2$ vertices. Hence,

$$\sum_{j=0}^{2n-2} r_j \leq (2n-2) + 3 \cdot (2n) + (2n-5) \cdot (2n-1) = 4n^2 - 4n + 3 < |K_{2n}| + 2\|G(1, 2n-3)\|, \rightarrow\leftarrow$$



$n = 4$

$\leftarrow G(1,5)$

Figure 48. $G(1,5)$.

On the other hand, if $\psi(u) \neq \psi(v)$, a similar argument shows that $2n-1$ colors are not enough. Hence, $G(1, 2n-3)$ is of Type 2. ■

(*) If G is of Type 1, then G is conformable.

Proof. If there exists a color i such that $|G| - |\psi_{V(G)}^{-1}(i)| \equiv 1$ (mod 2), then i occurs in at most $|G|-1$ vertices. Since every color occurs around a major vertex (Δ-degree), $\Delta(G)+1$ colors are not enough if $\operatorname{def}(G) < |\{ i \mid |G| - |\psi_{V(G)}^{-1}(i)| \equiv 1 \pmod 2\}|$. So, if G is not conformable, then G is not of Type 1. ■

Conjecture (Chetwynd and Hilton, 1988)

Let G be a simple graph with $\Delta(G) \geq \left\lfloor \frac{|G|+1}{2} \right\rfloor$. Then, G is of Type 2 if and only if there exists a non-conformable subgraph H of G such that $\Delta(H) = \Delta(G)$ (and G is not the Chen and Fu graph).
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(∘∘∘) The (original) conjecture was disproved by using Theorem 7.5.

(∘) The graph $G(1, 2n-3)$ is known as Chen and Fu graph.