FGT Lecture 10

Graph Labelings

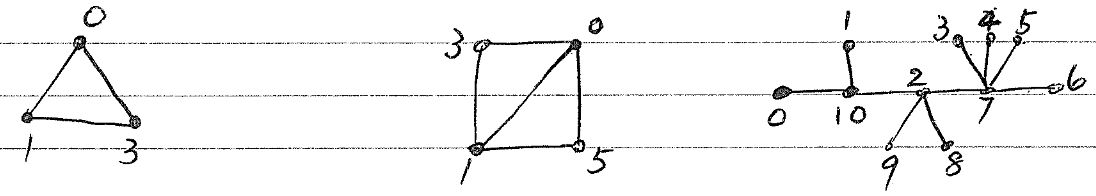
1. Graceful labeling

(∘) A graceful labeling of a graph G is a 1-1 mapping f,

$f: V(G) \xrightarrow{1-1} \{0, 1, 2, \cdots, \|G\|\}$ such that the weight of all of edges uv, defined by $|f(u) - f(v)|$, are distinct.

Examples

G :

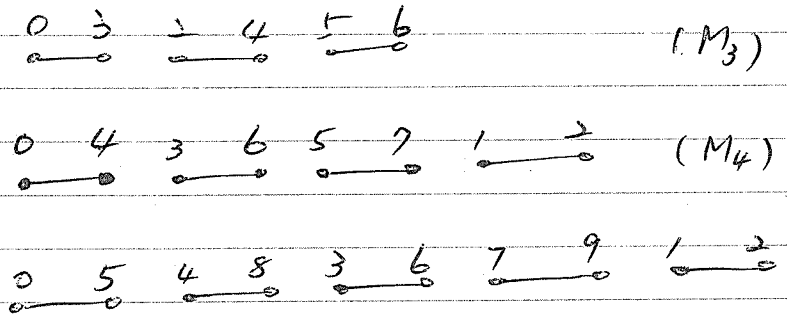


(∘∘) So, in general, we consider the graceful labeling of a connected graph since for tree $|G| = \|G\| + 1$. In order that f to be 1-1, $\|G\| \geq |G| - 1$.

(∘) If $\|G\| < |G|$, then, we use $f: V(G) \xrightarrow{1-1} \{0, 1, 2, \cdots, |G|\}$ instead.

Example

G:



(M_3)

(M_4)

(M_5)

(∘) A graph is graceful if it has a graceful (β) labeling.

β: Alex Rosa (1967)
Graceful: S. W. Golomb

Conjecture (Ringl-Kotzig) ~ 1980

Every tree is graceful.

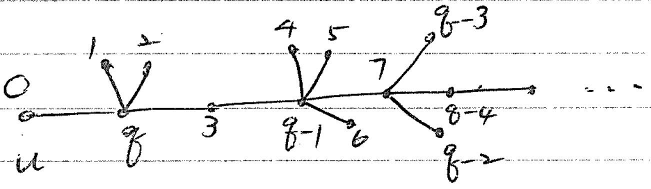
(*) Many results are obtained on special trees. But, no one can prove the conjecture so far.

Theorem

(1) Caterpillars are graceful.

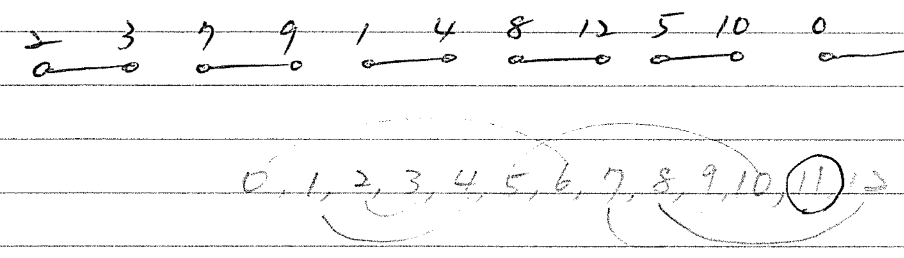
(2) M_n is graceful for all $n \in \mathbb{N}$.

Proof. (1) The proof can be obtained by a direct labeling: starting from the end vertex u as shown in the following figure, let its label be 0, and then its neighbor be q, ∘ ∘ ∘. Then, all weights are distinct.

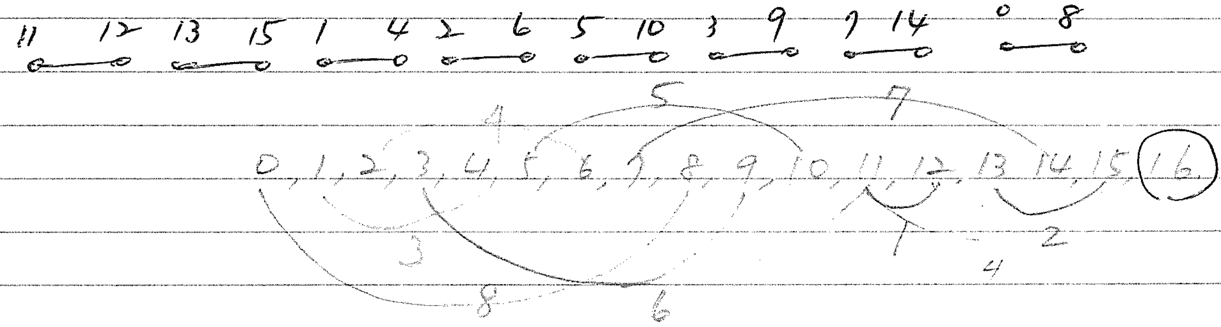


(2) The proof of this result is quite long.

The case when $n \equiv 0$ or $1 \pmod 4$ is known as Skolem sequence of type A and $n \equiv 2$ or $3 \pmod 4$ is Skolem sequence of type B. We shall give some examples here and omit the proof.



$n = 6$



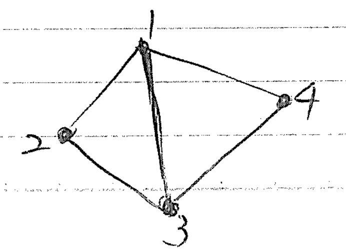
(○) In type A, 16 is not used and in type B, 11 is not used.

(*) Graceful labeling is very useful in constructing Combinatorial designs.

2. <u>Prime labeling</u>

(○) A prime labeling of a graph G of order n is a 1-1, onto mapping $f: V(G) \xrightarrow[\text{onto}]{1-1} \{1, 2, \cdots, n\}$ such that if $uv \in E(G)$, then $\gcd(f(u), f(v)) = 1$.

<u>Example</u>



Conjecture (Ringel).

Every tree has a prime labeling.

✓ (∘) Let G be a graph of order n. If $\alpha(G) < \lfloor \frac{n}{2} \rfloor$, then G has no prime labelings.

Proof. All even integers in $\{1, 2, \cdots, n\}$ form an independent set in a co-prime graph G_n.

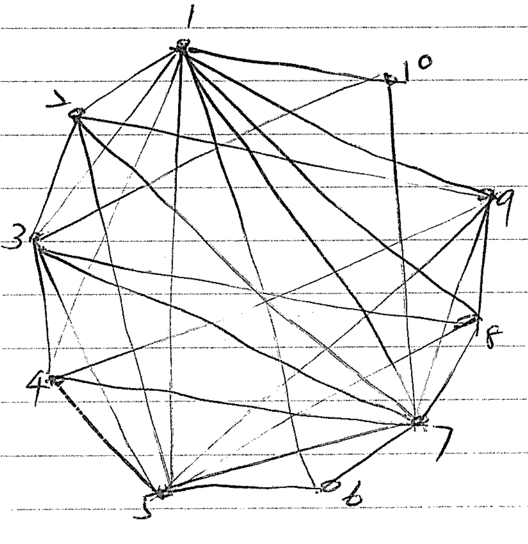
(*) Co-prime graph of order n, G_n.

$V(G_n) = \{1, 2, \cdots, n\} = [n]$.

$\{i, j\} \in E(G_n)$ if $i, j \in [n]$ and $\gcd(i, j) = 1$.

(∘∘) If a graph G of order n has a prime labeling, then $G \leq G_n$.

$n = 10$

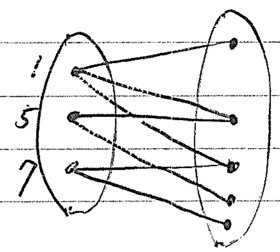


(*) For each n, there exists a prime in $(n, 2n)$. This is known as Bertrand's Postulate or sometimes Bertrand–Chebyshev Thm.

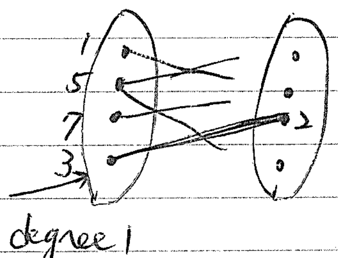
(*) The number of primes in $[n]$ is $\pi(n) \sim \frac{n}{\log n}$ (base e).

(∘) If T is a tree of order 8, then T has a prime labeling.

Proof. <u>Case 1</u> $T = (A, B)$ and $|A| \leq 3$.



<u>Case 2</u> $|A| = |B| = 4$



degree 1
(By Pigeon-hole principle)

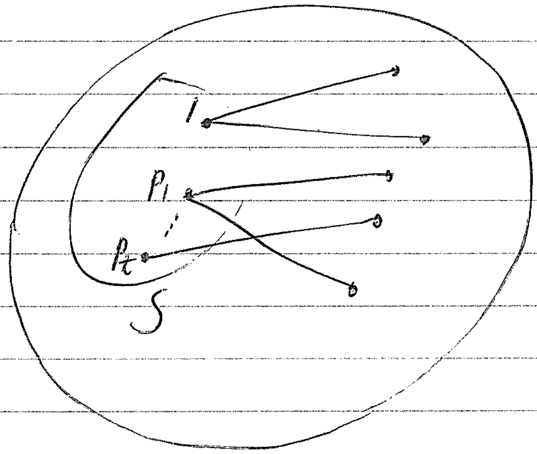
(∘) If G is a bipartite graph of order 8 and $G \ncong K_{4,4}$, then G has a prime labeling.

(∘) If G has a prime labeling and S is a set of edges in G, then $G - S$ has a prime labeling.

Let $P(a,b]$ denote the set of primes between integers a and b.

<u>Theorem</u> If G is a graph of order n and S is a vertex cover of G such that $|S| \leq P(\lceil \frac{n}{2} \rceil, n] + 1$, then G has a prime labeling.

<u>Proof</u>. Let S be labelled by 1, and primes in $P(\lceil \frac{n}{2} \rceil, n]$. Then, the proof follows. (Since S is a vertex cover, no edges exist in $\langle G-S \rangle_G$.) ▨



3. <u>Magic labelings</u> (on Edges)

G has a <u>magic labeling</u> if there exists a 1-1 mapping $f: E(G) \longrightarrow \{1, 2, 3, \cdots, \|G\|\}$ such that for each v, the weight of v, defined by $\sum_{u \in N_G(v)} f(uv)$, is a constant.

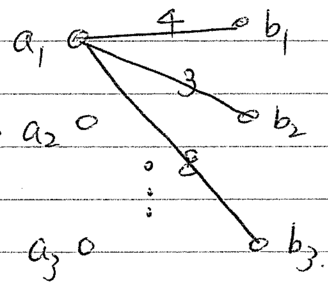
(◦) If all weights are different, then we have an <u>anti-magic labeling</u>.

Example For each $n=1$ and $n \geq 3$, $K_{n,n}$ has a magic labeling.

(◦) A magic labeling of $K_{n,n}$ is corresponding to the existence of a (weak) magic square (only rows and columns are required to be the same sum).

	b_1	b_2	b_3
a_1	4	3	8
a_2	9	5	1
a_3	2	7	6

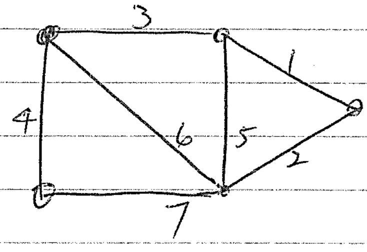
$\longleftrightarrow$



(•) For $n \neq 2, 6$, we can use a pair of orthogonal Latin squares to construct a magic labeling of $K_{n,n}$.

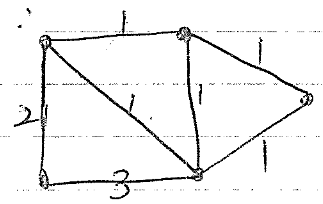
4. Anti-magic labelings

G has an anti-magic labeling if there exists a 1-1 mapping $f: E(G) \to \{1, 2, \cdots, \|G\|\}$ such that for each v, the weights $\sum_{u \in N_G(v)} f(uv)$ are all different.



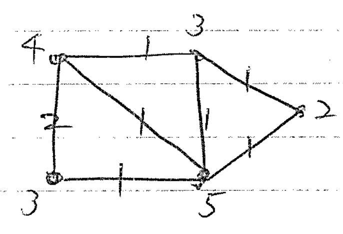
(○) We may also use $f : E(G) \longrightarrow \{1, 2, \cdots, k\}$ to obtain an

anti-magic labeling. $k \leq \|G\|$ in general



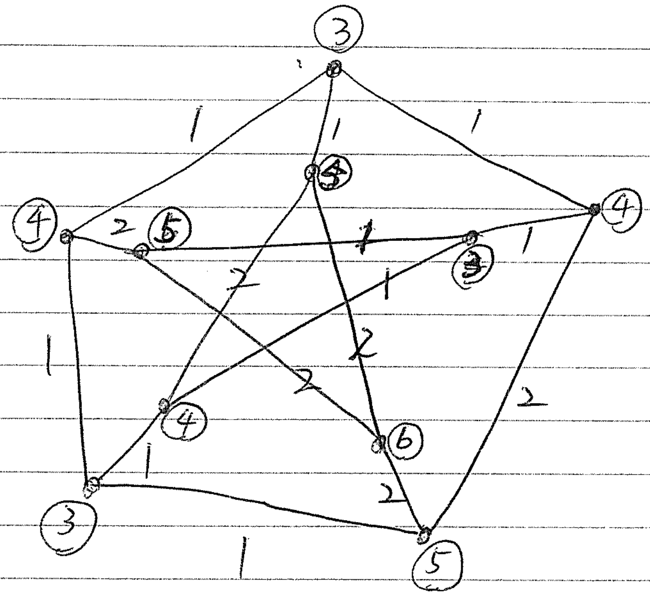
5. Vertex-distinguished labeling

We only require the weight of u is different from the weight of when uv is an edge of G.



Conjecture $k = 3$ is enough!

One more example:



Conjecture For all graphs G, a vertex-distinguished labeling can be obtained by a {1, 2, 3}-labeling. (Use 1, 2, 3 only!)

基礎圖論介紹到這裏，希望你(妳)有興趣，繼續學習。

Happy New Year!

Fu

2019, 1, 2.