Theorem 61 (Nordhaus and Gaddum, 1956)

If G is a graph of order p, then

(1) $2\sqrt{p} \underset{①}{\leq} \chi(G) + \chi(\overline{G}) \underset{①'}{\leq} p+1$, and

(2) $p \underset{②}{\leq} \chi(G) \cdot \chi(\overline{G}) \underset{②'}{\leq} \left[(p+1)/2\right]^2$.

Proof. First, we claim that ② $\chi(G) \cdot \chi(\overline{G}) \geq p$. For each vertex v of K_p, let $\varphi(v) = (\varphi_1(v), \varphi_2(v))$ where φ_1 and φ_2 are chromatic colorings of G and $\overline{G}$ respectively. Since two vertices of K_p are either adjacent in G or $\overline{G}$, all ordered pairs of $v \in V(K_p)$ are distinct. Hence, $\chi(G) \cdot \chi(\overline{G}) \geq p$. (We need p colors for K_p.)

This implies that $\frac{\chi(G)+\chi(\overline{G})}{2} \geq \sqrt{\chi(G) \cdot \chi(\overline{G})} \geq \sqrt{p}$, ① holds.

Now, let $k = \max_{H \leq G} \delta(H)$. We claim that every induced subgraph H' of $\overline{G}$ has minimum degree $p-k-1$, i.e. $\max_{H' \leq \overline{G}} \delta(H') \leq p-k-1$.

Suppose not. Let H'' be an induced subgraph of $\overline{G}$ such that $\delta(H'') = p-k$. Since H'' is an induced subgraph of $\overline{G}$, $H'' \cong \overline{H}$ for some induced subgraph H of G. Let $|H| = h$. Since $\delta(H'') = \delta(\overline{H}) = p-k$, $\deg_H(v) \leq (h-1)-(p-k)$ for each $v \in V(H)$.

Therefore, in G, $\deg_G(v) \leq (h-1)-(p-k)+(p-h) = k-1$. On the other hand, $k = \max_{H \leq G} \delta(H)$ and thus we have an induced subgraph $H''' \leq G$ such that $\delta(H''') = k$. This implies that $V(H) \cap V(H''') = \phi$. By the fact $|V(H''')| \geq k+1$, $|H| = h \leq p-(k+1)$ and thus $|\overline{H}| \leq p-(k+1) = p-k-1$. $\delta(\overline{H}) = p-k$ is not possible. This concludes that

$$\max_{H' \leq \overline{G}} \delta(H') \leq p-k-1 \text{ and thus } \chi(\overline{G}) \leq p-k-1+1 = p-k$$

(and $\chi(G) \leq 1+k$), the proof of ①$'$ follows.

Now, for ②$'$, it follows by

$$\sqrt{\chi(G)\cdot\chi(\overline{G})} \leq \frac{\chi(G)+\chi(\overline{G})}{2} \leq \frac{p+1}{2}.$$ ■

(•) A graph is said to be <u>self-complementary</u> if $G \cong \overline{G}$.

In this situation $\sqrt{p} \leq \chi(G) \leq \frac{p+1}{2}$. $p=5 \Rightarrow \chi(G) = 3$.
$\downarrow$
$G \cong C_5$

<u>Problem</u> Let $\omega(G)$ denote the order of a maximum clique, i.e., the order of complete subgraphs with maximum order. Then,

Clique number of G

$\chi(G) \geq \omega(G)$. When does the equaily holds?

(•) A graph G is called <u>perfect</u> if $\chi(H) = \omega(H)$ for each induced subgraph H of G. (*) $\chi(H) - \omega(H)$ can be very large!

<u>Theorem 62</u>

For every integer n, there exists a triangle-free graph G such that $\chi(G) = n$. $(\chi(G) - \omega(G) = n-2.)$

<u>Proof.</u> By induction on n and K_1, K_2, C_5 do have the property respectively for $n = 1, 2$ and 3. Now, assume that H is a triangle-free k-chromatic graph, i.e., $\chi(H) = k$. We construct a graph G based on H such that G is a triangle-free $(k+1)$-chromatic graph.

Let $V(H) = \{v_1, v_2, \dots, v_p\}$ and $V(G) = V(H) \cup \{u_1, u_2, \dots, u_p, u_0\}$.

Let $E(G) = E(H) \cup \{u_0u_i \mid i = 1, 2, \dots, p\} \cup \{u_iv_j \mid v_j \in N_H(v_i)\}$. See Figure 38 for an example when $k = 3$.

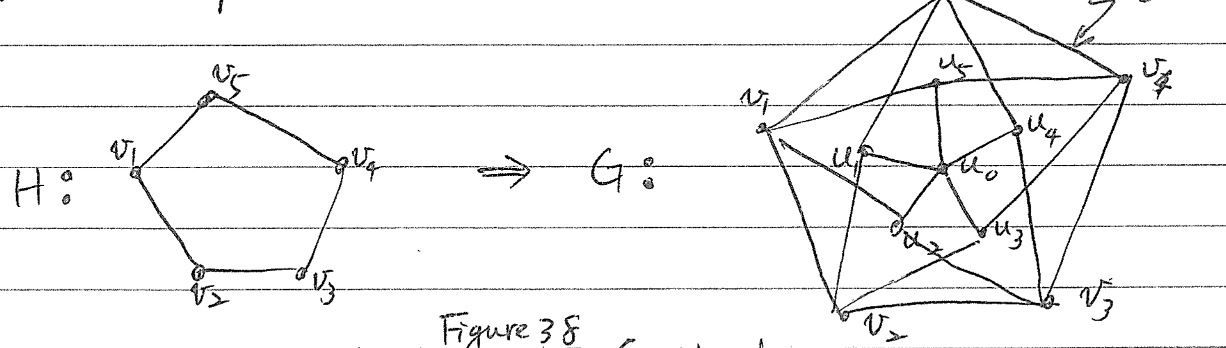

Figure 38

Grötzsch graph

Since $\langle\{u_1, u_2, \cdots, u_p\}\rangle_G$ contains no edges, u_0 is not in any triangle. By assumption, $H \not\cong K_3$. So, the only possibility will a triangle consists of u_i, v_j and v_k where $u_i v_j$ and $u_i v_k$ are edges of G. If they form a triangle, then $\langle\{v_i, v_j, v_k\}\rangle_H$ is a triangle in H. Hence, G is triangle-free.

Now, we claim $\chi(G) = k+1$. Let φ be a k-coloring of H. Let $\tilde{\varphi}: V(G) \to \{1, 2, \cdots, k+1\}$ by letting $\tilde{\varphi}(u_i) = \varphi(v_i)$ and $\tilde{\varphi}(u_0) = k+1$. Hence, we have a $(k+1)$-coloring of G, thus $\chi(G) \le k+1$. On the other hand, we show that $\chi(G) \ge k+1$. Suppose not. Let φ' be a k-coloring of G and the colors used are $1, 2, \cdots, k$. First, we assign u_0 the color k, i.e., $\varphi'(u_0) = k$. So, the colors used for $u_1, u_2, \cdots, u_p$ must be in $\{1, 2, \cdots, k-1\}$. Since $\chi(H) = k$, k occurs somewhere in H, say v_i. (May have more vertices.) Now, we recolor v_i by using $\varphi'(u_i)$. Since u_i is adjacent to every vertex of $N_H(v_i)$, $\varphi'(u_i) \ne \varphi'(v)$ for each $v \in N_H(v_i)$ and thus we have a proper coloring of H using at most $k-1$ colors. (?) $\to\leftarrow$ ∎

(*) This theorem has been extended to a more general result obtained by Erdös and Lovász (1961): For any integers $m, n \geq 2$, there exists an n-chromatic graph whose girth exceeds m. (Theorem 62 is for $m=3$.)

** Theorem 63 (Lovász, 1972) (Weakly Perfect Graph Theorem)

A graph G is perfect if and only if $\overline{G}$ is perfect.

Note. The proof of this Theorem is not too long. But, the proof of next one is long.

*** Theorem 63. (Maria Chudnovsky, Neil Robertson, Paul Seymour and Robin Thomas, Annals of Mathematics, 164(2006), 51–229.)

A graph G is perfect if and only if no induced subgraph of G or $\overline{G}$ is an odd cycle of length at least 5.

Proof of Theorem 63

We prove a different version: (Stronger)

A graph G is perfect if and only if $|H| \leq \alpha(H)\cdot\omega(H)$ (1)

for all induced subgraphs H of G. ($\omega(H)$ is the clique number of H.)

(∘) In $\overline{G}$, if A is an independent set, then in G, $\langle A\rangle_G$ is a clique
$\langle A\rangle_{\overline{G}}$ (a clique) (A is independent)

($\Rightarrow$) If G is perfect, then for each induced subgraph H, $\chi(H)=\omega(H)$.
(Definition)
Hence, the vertex set of H, $V(H)$, can be partitioned into $\omega(H)$ subsets.

Clearly, each subset has size at most $\alpha(H)$, hence $|H| \le \alpha(H)\cdot\omega(H)$.

($\Leftarrow$) By induction on $|G|$. Assume that every induced subgraph H of G satisfying (1), but G is not perfect. (Every "proper" induced subgraph is perfect.)

Now, let $u \in V(G)$ and consider $G-u$. Let $\omega(G)=\omega$ and $\alpha(G)=\alpha$. By induction, $\chi(G-u)=\omega(G-u)$. If $\omega(G-u)<\omega(G)$, then by coloring u with a new color, we have $\chi(G) \le \omega(G)$. This implies that G is perfect. $\rightarrow\leftarrow$ (We can replace u with an independent set!)

Let K be the vertex set of a clique with ω vertices. Notice that if $u \notin K$, then K meets every color class (independent set) of $G-u$. (2) But, if $u \in K$, then K meets $\omega-1$ color classes of $G-u$. (3)

Now, we construct $\alpha\omega+1$ independents in G by the followings. Let $A_0=\{u_1,u_2,\cdots,u_\alpha\}$ be an independent set of G with α vertices (independence number α).

Starting from $G-u_1$, and then $G-u_2, \cdots, G-u_\alpha$, we have $\alpha\omega$ independent sets : $A_1, A_2, \cdots, A_\omega, A_{\omega+1}, A_{\omega+2}, \cdots, A_{2\omega}, \circ \circ \circ, A_{\alpha\omega}$. (Each of them contains ω independent sets.)

Observe that $K \cap A_i = \phi$ for all but one $i \in \{0, 1, 2, \cdots, \alpha\omega\}$.

(If $K \cap A_0 = \phi$, then $K \cap A_i \neq \phi$ for all $i \in \{1, 2, \cdots, \alpha\omega\}$ (by (2)). On the other hand, if $K \cap A_0 \neq \phi$, then $|K \cap A_0| = 1$, say $K \cap A_0 = \{u_j\}$.
(Except for u_j, all the other vertices of A_0 are not in K.)
This implies that K meets $\omega - 1$ color classes of $G - u_j$ which implies that in $G - u_j$, there is an independent set A_i such that $K \cap A_i = \phi$.
(By (3).))

Finally, let M and N be defined as in Figure 39.
((0,1)-matrices)

Let $V(G) = \{v_1, v_2, \cdots, v_p\}$. Let $\boxed{K_i \subseteq G - A_i}$ for each $i = 0, 1, \cdots, \alpha\omega$.
(?)

$$M : \begin{array}{c} \\ A_0 \\ A_1 \\ \vdots \\ A_{\alpha\omega} \end{array} \begin{array}{c} v_1 \; v_2 \; \cdots \; v_p \\ \left[\begin{array}{c} \\ M(i,j) = 1 \\ \text{iff } v_j \in A_i \\ \\ \end{array} \right]_{(\alpha\omega+1) \times p} \end{array} \qquad N : \begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_p \end{array} \begin{array}{c} K_0 \; K_1 \; \cdots \; K_{\alpha\omega} \\ \left[\begin{array}{c} \\ N(i,j) = 1 \\ \text{iff } v_i \in K_j \\ \\ \end{array} \right] \end{array}$$

(?) By induction $\chi(G - A_i) = \omega(G - A_i) = \omega(G)$.
otherwise, G is perfect.

Since $M \cdot N = J_{(\alpha\omega+1)\times(\alpha\omega+1)} - I_{\alpha\omega+1}$ is non-singular, the rank of $M \cdot N$ is $\alpha\omega+1$ which is larger than $p = |G|$, a contradiction to the assumption when $H \cong G$. Hence, the proof follows. ∎

Theorem 64 If G is a connected planar graph, then $\chi(G) \leq 5$.

Proof. By induction on $|G|$. By Theorem 57, it suffices to consider an induced subgraph H whose minimum degree is 5.

Let $v \in V(H)$ such that $\deg_H(v) = 5$. By induction, $\chi(H-v) \leq 5$. Let φ be a 5-coloring of H and we consider the colors assigned on $N_H(v)$. Let them be $\varphi(v_1), \varphi(v_2), \cdots, \varphi(v_5)$. Clearly, if any two of them are of the same color, then there is a color for v such that we have a proper 5-coloring of H. So, assume that $\varphi(v_i) = i$, $i = 1, 2, 3, 4, 5$ and the vertices are in clockwise order, see Figure 40.

Now, consider the induced subgraph $H_{1,3} = \langle \varphi^{-1}(1) \cup \varphi^{-1}(3) \rangle_H$. If v_1 and v_3 are in distinct components, then by changing the colors 1 and 3 in the component which contains v_1, we obtain a new coloring such that $\varphi(v_1) = 3$ and $\varphi(v_3) = 3$. Hence, 3 is available for v.

On the other hand, there exists a path P connecting v_1 and v_3. Hence, $v - v_1 - P - v_3 - v$ is a cycle such that v_2 and v_4 are in different regions. By a similar argument, we may change the color of v_2 to 4. Then, 2 is available for v. ■

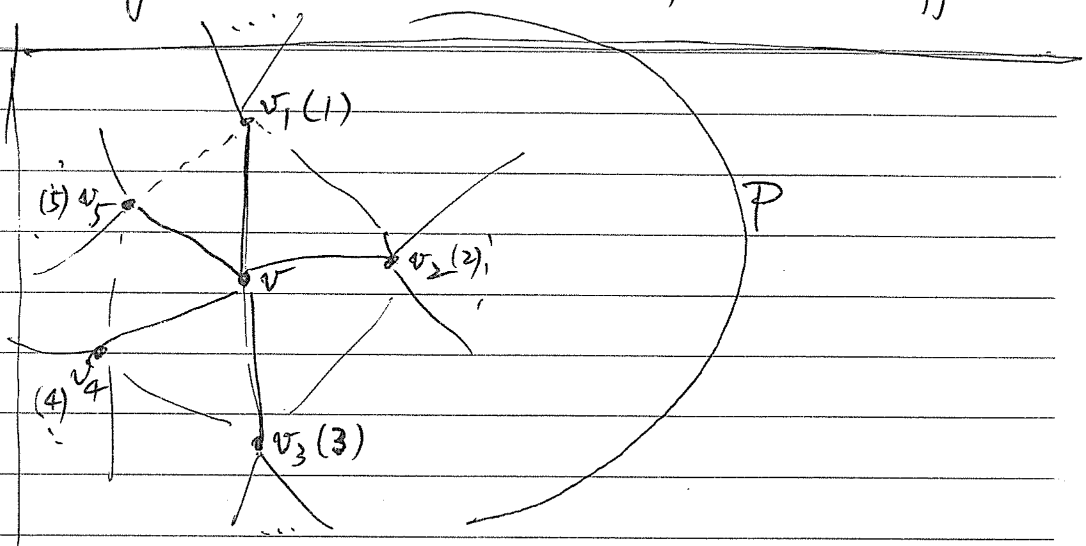

Figure 40

(***) Theorem (4CT) Every planar graph is 4-colorable.

The most recent proof was obtained by N. Robertson, D.P. Sanders, P.D. Seymour and R. Thomas (1996): A new proof of the 4CT, Electron. Res. Announc. A.M.S. 2, 17–25.

The first proof was obtained in $\overset{1976-}{1977}$, by K. Appel and W. Haken.

(*) The following theorem is not working for $n=0$.

Theorem 65 (The Heawood Map Coloring Theorem)

For every positive integer n, $\chi(S_n) = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$.

($\chi(S_n)$: the maximum chromatic number among all graphs that can be embedded on S_n.)

Proof. (Outline)

The upper bound $\chi(S_n) \le \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$ was obtained by Heffter in 1890. At that time, he claimed that it's an equality. But, unfortunately, the correct proof came out many years later by the effort of considering the embedding of K_p since for sure K_p needs p colors.

So, define $p = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$. It follows that $n \ge (p-3)(p-4)/12$ and thus $n \ge \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$. By the fact $\gamma(K_p) = \left\lceil \frac{(p-3)(p-4)}{12} \right\rceil$, $\chi(S_n) \ge p = \left\lfloor \frac{7+\sqrt{1+48n}}{2} \right\rfloor$. ∎