Graph Theory Lecture 14

(•) There exists a procedure to construct all the graphs G with $\chi(G) \geq k$.

(••) $\mathcal{G}_k$ is a collection of k-constructible graphs if $G \in \mathcal{G}_k$ can be constructed recursively by the following steps.

(i) K_k is k-constructible.

(ii) If G is k-contructible and $x, y \in V(G)$ are non-adjacent, then $(G+xy)/xy$ is k-constructible.

(iii) If G_1 and G_2 are k-constructible such that $V(G_1) \cap V(G_2) = \{x\}$, $xy_1 \in E(G_1)$, $xy_2 \in E(G_2)$, then $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible, see Figure 41.

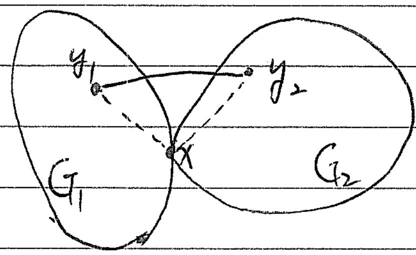


Figure 41. Hajós construction.

(*) If G is k-constructible, then $\chi(G) \geq k$.

(i) is trivial. (ii) If $(G+xy)/xy$ uses less than k colors, then by coloring x and y with the same color, we have $\chi(G) < k$.

(iii) If the graph obtained uses less than k colors, then either $\chi(G_1) < k$ or $\chi(G_2) < k$ depending on whether $\varphi(y_1) \neq \varphi(x)$ or $\varphi(y_2) \neq \varphi(x)$. Since y_1 and y_2 receive distinct colors, one of the above two conditions must hold.

<u>Theorem 66.</u> (Hajós, 1961)

Let G be a graph. Then, $\chi(G) \geq k$ if and only if G has a k-constructible subgraphs.

<u>Proof.</u> ($\Leftarrow$) It has been explained above.

($\Rightarrow$) Suppose not; then $k \geq 3$. Let G be a <u>maximal</u> counterexample, i.e. G is of maximum size such that G does not contain a k-constructible subgraph. Now, G can not be a complete r-partite graph. For otherwise, $\chi(G) \geq k$ implies that $r \geq k$ and then G contains a k-constructible K_k. (Contract each partite set.) Hence there exist vertices x, y_1 and y_2 such that $xy_1 \notin E(G)$, $xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. By assumption of the maximality of G, both $G+xy_1$ and $G+xy_2$ contain k-constructible subgraphs, say H_1 and H_2;

moreover, $xy_1 \in E(H_1)$ and $xy_2 \in E(H_2)$.

Let $H_2 - H_1$ denote the graph $\langle V(H_2) \setminus V(H_1) \rangle_{H_2}$ and H_2' is an isomorphic copy of H_2 such that $V(H_2') \cap V(G) = \{x\} \cup (V(H_2) \setminus V(H_1))$, see Figure 42. So, $V(H_1) \cap V(H_2') = \{x\}$. Now, since $H_2' \cong H_2$, let $\varphi : H_2 \to H_2'$ be an isomorphism. By (iii) $H_1 \cup H_2' - xy_1 - x\cdot\varphi(y_2) + y_1\cdot\varphi(y_2)$ is k-constructive, let this graph be H. Now, for each vertex v' in $V(H_2') \setminus V(G)$, there exists a v such that $v' = \varphi(v)$. Furthermore vv' is not an edge of H. By (ii), we can identify v and v' and obtain a k-constructive subgraph of G after identifying all vertices in $V(H_2) \setminus V(H_1)$. ■

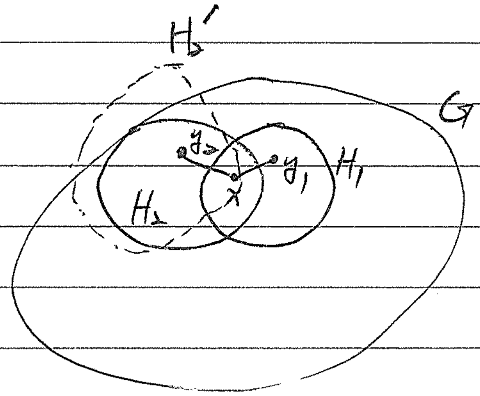


Figure 42

(∘) A k-edge-coloring is a mapping $\pi: E(G) \to \{1, 2, \cdots, k\}$ such that incident edges receive distinct images (colors).

(∘) $\chi'(G) = \min.\{k \mid G$ has a k-edge-coloring$\}$. (Chromatic index of G.) If $\chi'(G) = k$, then G is h-edge-colorable for each $h \geq k$.

Theorem 67. (Vizing, 1964)

If G is a simple graph, then $\Delta(G) \leq \chi' \leq \Delta(G)+1$.

Proof. The left hand inequality is easy to see, we prove the right hand inequality. By induction on $\|G\|$. We shall prove that G has a $\underset{(\text{coloring in short})}{\underline{(\Delta(G)+1)\text{-edge-coloring}}}$ for G and the assertion is true for smaller sizes, i.e., for each $e \in E(G)$, $G-e$ has a coloring π.

First, we observe that since each vertex v is of degree at most $\Delta(G)$, a color is missing around v. Second, if α and β be two colors used in the (a) coloring, then α and β induce a subgraph with components either paths or even cycles.

Finally, if G has no colorings using $\Delta(G)+1$ colors, then for each edge xy and any coloring of $G-xy$, there exists an α-β path from y ends in x provided α is missing at x and β is missing at y. See Figure 43 for missing colors.

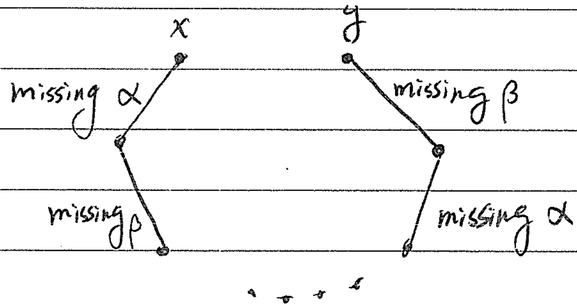


Figure 43

(*) If α-β path does not connect x and y, then we may recolor one of the path (α,β), to obtain a coloring of G using $\Delta(G)+1$ colors.

Clearly, if x and y are missing the same color, then we can use that color to color xy and obtain a $\Delta(G)+1$ coloring of G.

Claim: There is a way to recolor some edges in $G-xy$ such that x and y miss the same color.

Outline of proof

Let $M(y)$ denote the colors missing at y, and $c_1 \in M(y)$. Now, consider $M(x)$. If $c_1 \in M(x)$, then color xy by c_1 results in a $\Delta(G)+1$ coloring of G. (The claim holds.)

Hence, $c_1 \notin M(x)$, let $c_0 \in M(x)$ and $\pi(xy_1) = c_1$. (See Figure 44)

Then, consider $M(y_1)$ and let $c_2 \in M(y_1)$. Note that $c_2 \notin M(x)$.

If $c_2 \in M(x)$, then we let $\pi(xy_1) = c_2$. Thus, c_1 becomes a missing color in $M(x)$, the coloring for xy is available, $\pi(xy) = c_1$. This fact will continue: $c_2 \notin M(x) \Rightarrow \exists\, y_2$, s.t. $\pi(xy_2) = c_2$; and then $c_3 \in M(y_2)$, $\pi(xy_3) = c_3$, $\circ\circ\circ$, $c_{i+1} \in M(y_i)$, $\pi(xy_{i+1}) = c_{i+1}$.

Since we only have $\Delta(G)+1$ colors, there exists an l such that $\pi(xy_{l+1}) = c_{l+1} \in \{c_1, c_2, \cdots, c_l\}$. W.L.O.G., let $c_{l+1} = c_k$, $k \in \{1, 2, \cdots, l\}$.

Now, we have several cases to consider depending on whether $c_0 \in M(y_l)$ or $c_0 \notin M(y_l)$.

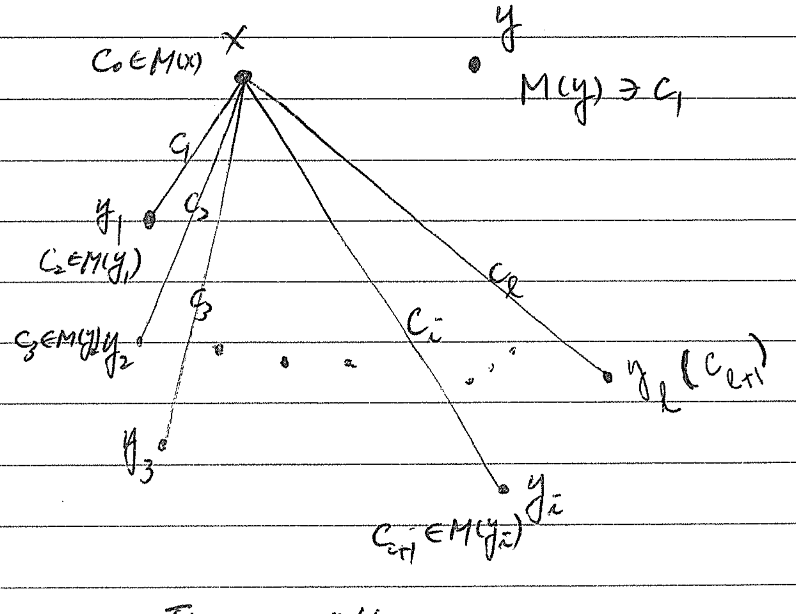


Figure 44

(a) $c_0 \in M(y_\ell)$.

Then, let $\pi(xy_\ell) = c_0$, $\pi(xy_{\ell-1}) = c_\ell$, $\cdots$, $\pi(xy_1) = c_2$. This implies that $c_1 \in M(x)$ and the proof follows.

(b) $c_0 \notin M(y_\ell)$.

Since $c_{\ell+1} = c_k$, $c_k \in M(y_\ell)$. Now, consider $c_k - c_0$ path starting from y_ℓ.

(i) It is a $y_\ell - y_k$ path. Since $\pi(xy_k) = c_k$, we may recolor them to a $c_0 - c_k$ path starting from y_k. (Note here that c_0 occurs in an edge incident to y_ℓ. By the fact that the last color is c_k, both c_0 and c_k occur an even number of times.) Now, since $\pi(xy_k) = c_0$, the recoloring of $xy_1, xy_2, \cdots, xy_{k-1}$ gives $c_1 \in M(x)$, we have the proof.

(ii) It is a $y_\ell - y_{k-1}$ path. Since $c_k \in M(y_{k-1})$, this path is ended with color c_0. That is to say c_0 is also available for xy_{k-1} (not only c_{k-1}). Hence, we color xy_{k-1} with c_0 instead of c_{k-1}, the proof follows by a similar recoloring process.

(iii) It is a $y_e - y_i$ path, $i \notin \{k-1, k\}$.

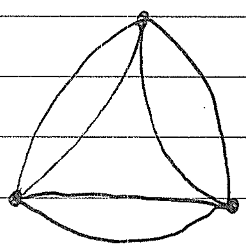
Then, either c_e or c_0 will be available for xy_i and the proof follows by recoloring process. ■

(◦) Based on the same proof technique, we also have a stronger result of Vizing's Theorem.

Theorem 67′ (Vizing, 1964)

If G is a multigraph with multiplicity η, then $\chi'(G) \leq \Delta(G) + \eta$.

(◦) The following graph has $\Delta(G) = 4$ and $\eta = 2$.



Definition (Class 1 and Class 2)

A graph (simple) is of Class 1 if $\chi'(G) = \Delta(G)$ and of Class 2 if $\chi'(G) = \Delta(G) + 1$.

Theorem 68 (König, 1916) A bipartite graph is of Class 1.

Proof. (1st)

By induction on $\|G\|$. Let $xy \in E(G)$ and $G - xy$ can be edge-colored with $\Delta(G)$ colors. Now, since $\deg_{G-xy}(x) < \Delta(G)$ and $\deg_{G-xy}(y) < \Delta(G)$, a color is missing at x and also a color is missing at y. Let them be α and β respectively. Clearly, $\alpha \neq \beta$, and β occurs around x and α occurs around y. Now, we adapt the idea in proving Vizing's Theorem, let P be a longest α-β path from x : $\overset{\beta}{\bullet\!\!-\!\!\bullet}\overset{\alpha}{-\!\!\bullet}\overset{\beta}{-\!\!\bullet}\cdots$.

First, if P is an x-y path and the last edge has color α, then P is a path of even length. Hence, $P \cup \{xy\}$ is an odd cycle. $\rightarrow\leftarrow$ G is bipartite

Hence, x and y are in different components induced by the set of edges colored α and β. Now, we recolor all the edges of P by interchanging α and β. This gives a coloring in which β is missing at x and also at y. By coloring xy with β, we obtain a Δ-edge-coloring of G. ∎

<u>2nd proof</u>

<u>Lemma</u> Let G be a bipartite graph. Then, there exists a $\Delta(G)$-regular bipartite graph $\tilde{G} \geq G$. (Exercise)

By Lemma $\tilde{G}$ is a $\Delta(G)$-regular bipartite graph and thus $\tilde{G}$ can be decomposed into $\Delta(G)$ perfect matchings by König's Theorem. This implies that $\chi'(\tilde{G}) = \Delta(G)$. Since $G \leq \tilde{G}$, $\chi'(G) \leq \chi'(\tilde{G}) \leq \Delta(G)$. Hence, we conclude the proof.

(∘) A graph G is said to be overfull if $\|G\| > \lfloor \frac{|G|}{2} \rfloor \cdot \Delta(G)$.

(∘∘) If G is overfull, then G is of Class 2.

(∘) If G is overfull, then $|G|$ is odd.

Theorem 69

Petersen graph is of Class 2.

Proof. If G is the Petersen graph and $\chi'(G) = 3$, Then G can be decomposed into 3 1-factors: F_1, F_2 and F_3 (3 color classes).

Now, consider the set of 5 link-edges e_1, e_2, e_3, e_4 and e_5, see Figure 45.

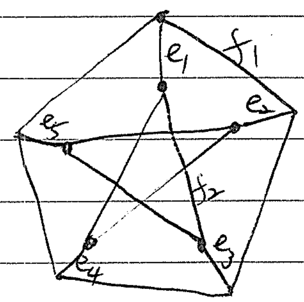


Figure 45 Petersen graph

At least one of F_1, F_2 and F_3 will contain at least two link-edges by Pigeon-hole principle, let it be F_1. Clearly, F_1 can not contain all the 5 link-edges. For otherwise, two C_5's is the union of F_2 and F_3 which is impossible. So, there are three cases to consider.

(i) $|F_1 \cap \{e_1, e_2, \cdots, e_5\}| = 4$

Let e_1 be the edge not in F_1. But, now all the edges of $G-e_1$ not in $\{e_2, e_3, e_4, e_5\}$ are incident to an edge of $\{e_2, e_3, e_4, e_5\}$. So, no other edge can be chosen for F_1.

(ii) $|F_1 \cap \{e_1, e_2, \cdots, e_5\}| = 3$

Let e_1 and e_2 be the edges not in F_1. Then, other than link-edges, we can choose at most one more edge f_1. The case e_1 and e_3 are not in F_1 has similar conclusion (only f_2 is available).

(iii) $|F_1 \cap \{e_1, e_2, \cdots, e_5\}| = 2$

This case comes out that we can find two more edges which not link-edges. ∎

(*) The proof of Theorem 69 implies that the Petersen graph contains no <u>Hamilton cycles.</u>

Proof. If G contains a Hamilton cycle C, then $\chi'(G)=3$ by coloring the cycle with two colors and $G-C$ (1-factor) with another color. ∎

<u>Theorem 70</u>

A 3-regular planar graph G is of Class 1.

<u>Proof.</u> Let G be embedded in S_0. Then, by 4-color Theorem, G is 4-face-colorable (or 4-map-colorable). Let the 4 colors used be obtained from the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, \oplus)$. Since each edge is in the boundary of two adjacent faces, let the edge be colored by $(a_1,b_1)\oplus(a_2,b_2)$ where (a_1,b_1) and (a_2,b_2) are the colors of these two adjacent faces. As a conclusion, we obtain a 3-edge-coloring of G, since $(0,0)$ will not be used. The coloring is proper since three adjacent faces will receive three different colors, see Figure 46. ∎

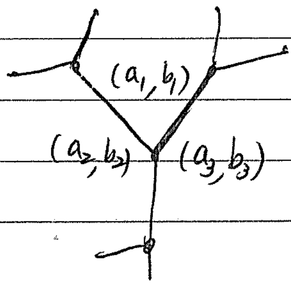


Figure 46.