Graph Theory Lecture 12 Nov. 28 —

Vertex-Coloring

(∘) k-coloring (proper) : $\varphi : V(G) \to \{1, 2, 3, \cdots, k\}$ s.t.
$uv \in E(G) \Rightarrow \varphi(u) \neq \varphi(v)$.

(∘) $\chi(G) = \min\{k \mid G \text{ has a } k\text{-coloring}\}$ (Chromatic number of G)

(∘) G is n-critical (chromatically) if $\chi(G-v) < \chi(G)$ for each $v \in V(G)$.

(∘∘) Every graph G has an n-critical induced subgraph H.

Theorem 56

Every critically n-chromatic graph (n-critical), $n \geq 2$, is $(n-1)$-edge-connected.

Proof. First, if $n=2$, then $G \cong K_2$ and thus G is 1-edge-connected.
If $n=3$, then $G \cong C_{2m+1}$, $m \geq 1$, (?) and G is 2-edge-connected.
Let $n \geq 4$ and assume that G is not $(n-1)$-edge-connected.
Hence, $V(G) = V_1 \cup V_2$ such that $|\langle V_1, V_2 \rangle| < n-1$ $(\leq n-2)$. Let $G_1 = \langle V_1 \rangle_G$
and $G_2 = \langle V_2 \rangle_G$. Now, both of them are $n-1$ colorable since

Now, consider the vertices incident to the edges in $\langle V_1, V_2 \rangle$. If for each edge $uv \in \langle V_1, V_2 \rangle$, $\varphi_1(u) \neq \varphi_2(v)$, then G has an $(n-1)$-coloring, a contradiction. Thus, assume that for some edges $uv \in \langle V_1, V_2 \rangle$, $\varphi_1(u) = \varphi_2(v)$. (We shall $\underbrace{\text{permute the colors of } G_1}_{\varphi_1'}$ in order that for each $uv \in \langle V_1, V_2 \rangle$, $\varphi_1'(u) \neq \varphi_2(v)$.

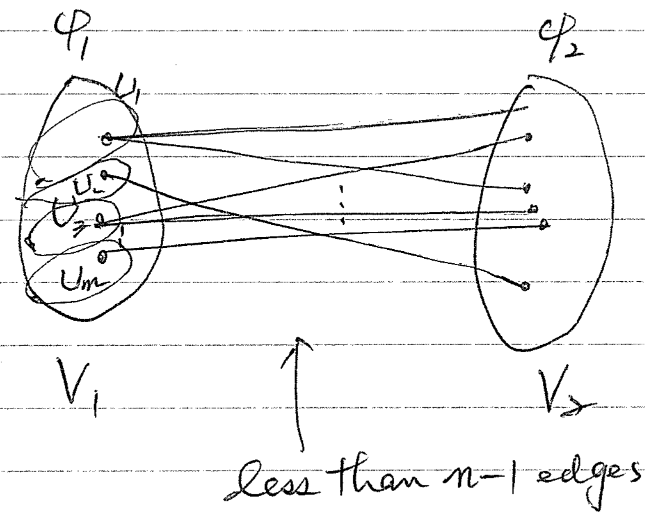


Figure 3 7.

Let $U_1, U_2, \cdots, U_m$ be the subsets of V_1 such that $\varphi_1^{-1}(i) = U_i$, $i = 1, 2, \cdots, m \ (\leq n-2)$ and there is at least one edge joining U_i and $V(G_2)$ for each i. Furthermore, let n_i be the number of vertices in U_i which are incident to a vertices of $V(G_2)$. Hence, $\sum_{i=1}^{m} n_i \leq n-2$.

Now, we start a process to recolor the vertices in V_1. Starting U_1. If $\forall\ x \in U_1$, $\varphi_1(x)$ has distinct colors with the colors of those vertices in V_2 which are incident to U_1, then we go to consider U_2. Otherwise, $\varphi_1(x) = \varphi_2(y)$ for some $x \in U_1$, and $xy \in \langle V_1, V_2 \rangle$. In this case, we permute the colors of $U_1, U_2, \cdots, U_m$ (It was $1, 2, \cdots, m$.) such that the color used for the vertices in U_1 is distinct from the colors of vertices in V_2 which are incident to U_1, there are n_1 of them. Since $\sum_{i=1}^{m} n_i \le n-2$, $n-1-n_1 > 0$ and thus there exists a color for U_1.

Following this idea, we consider U_2. If there are vertices x in U_2 such that $\varphi_1(x) = \varphi_2(y)$ for some $xy \in \langle V_1, V_2 \rangle$, then permute the colors used in $U_2, U_3, \cdots, U_m$ where the color for U_1 is fixed. Again, since $n-2-n_2 \ge (n-1)-n_1-n_2 > 0$, a color for U_2 is available. Continuing this process, we end it up with an $(n-1)$-coloring of G, a contradiction to $\chi(G) = n$. ∎

(c) If G is n-critical, then $\delta(G) \ge n-1$.

Theorem 57.

Let $k = \max_{H \preceq G} \delta(H)$. Then, $\chi(G) \leq k+1$.

Proof. (1st) Let $\chi(G) = n$ and H' be an n-critical induced subgraph of G. Then, $\delta(H') \geq n-1$. Since $\max_{H \preceq G} \delta(H) \geq \delta(H') \geq n-1$, $k \geq n-1$ and thus $\chi(G) = n \leq k+1$. ∎

2nd proof. (G is a (p,q)-graph.)

Since $k = \max_{H \preceq G} \delta(H)$, $\delta(G) \leq k$. Let $x_p \in V(G)$ and $\deg_G(x_p) = \underset{\leq k}{\delta(G)}$. Moreover, let $G_{p-1} = G - x_p$. Again, G_{p-1} has a vertex of degree at most k. So, we obtain a sequence of induced subgraphs, $G = G_p \geq G_{p-1} \geq \cdots \geq G_1$ such that $\delta(G_i) \leq k$ for $i = p, p-1, \cdots, 1$ such that $x_i \in V(G_i)$. Hence, we obtain a sequence $\langle x_1, x_2, \cdots, x_p \rangle$ such that x_{j+1} is incident to at most k vertices in $\langle \{x_1, x_2, \cdots x_j\} \rangle_G$.

This implies that we can use greedy algorithm to color G starting from x_1, and then $x_2, \cdots, x_p$. All we need is at most $k+1$ colors.

Hence, $\chi(G) \leq k+1$. ∎

Theorem 58 (Brooks, 1941)

Let G be a connected graph which is neither a complete graph nor an odd cycle. Then, $\chi(G) \leq \Delta(G)$.

Proof. By induction on $|G| = p$. We may assume that the graph we consider is 2-connected and Δ-regular where $\Delta \geq 3$. (?) (Note that a 2-regular connected graph G with $\chi(G) = 3$ is an odd cycle.)

First, if G is 3-connected, let x_p be any vertex such that $\langle N_G(x_p) \rangle_G$ is not a complete subgraph of G. Such an x_p does exist since G is not a complete graph. Let x_1 and x_2 be two vertices in $N_G(x_p)$ such that $x_1 \not\sim_G x_2$. Now, we may construct a sequence corresponding to $V(G)$. Choose $x_{p-1} \in \underset{\text{(3-connected)}}{N_G(x_n) \setminus \{x_1, x_2\}}$. Then, x_{p-2} is adjacent to either x_n or x_{n-1}. As a consequence, we have a sequence $\langle x_1, x_2, \cdots, x_p \rangle$ such that x_i is incident to at least one vertex in $\{x_{i+1}, x_{i+2}, \cdots, x_p\}$. Now, we use the greedy algorithm

Second, let G be 2-connected (but not 3-connected), i.e. $\kappa(G)=2$. Let S be a cut set with two vertices and $x_p \in S$. Hence, $G - x_p$ has a cut vertex, see Figure 38. Let x_1 and x_2 be two vertices in distinct blocks (2-connected maximal subgraph of G). Again, we use the idea mentioned above to construct a sequence $\langle x_1, x_2, \cdots, x_n \rangle$ and the proof follows by using the greedy algorithm for vertex coloring. ∎

(◦) $\Delta(G) - \chi(G)$ can be arbitraily large.

(◦) There are also graphs such that $\Delta(G) = \chi(G)$, for example even cycles, non-bipartite 3-regular graphs, say, Petersen graph.

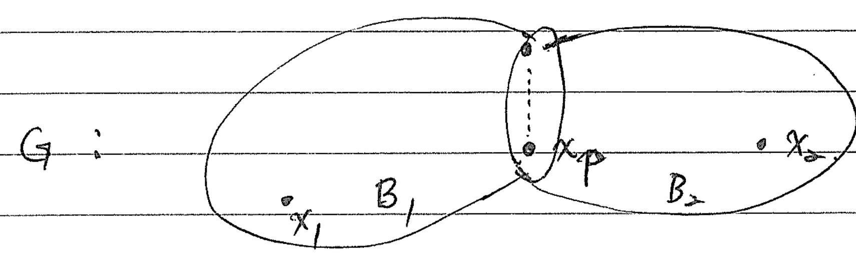
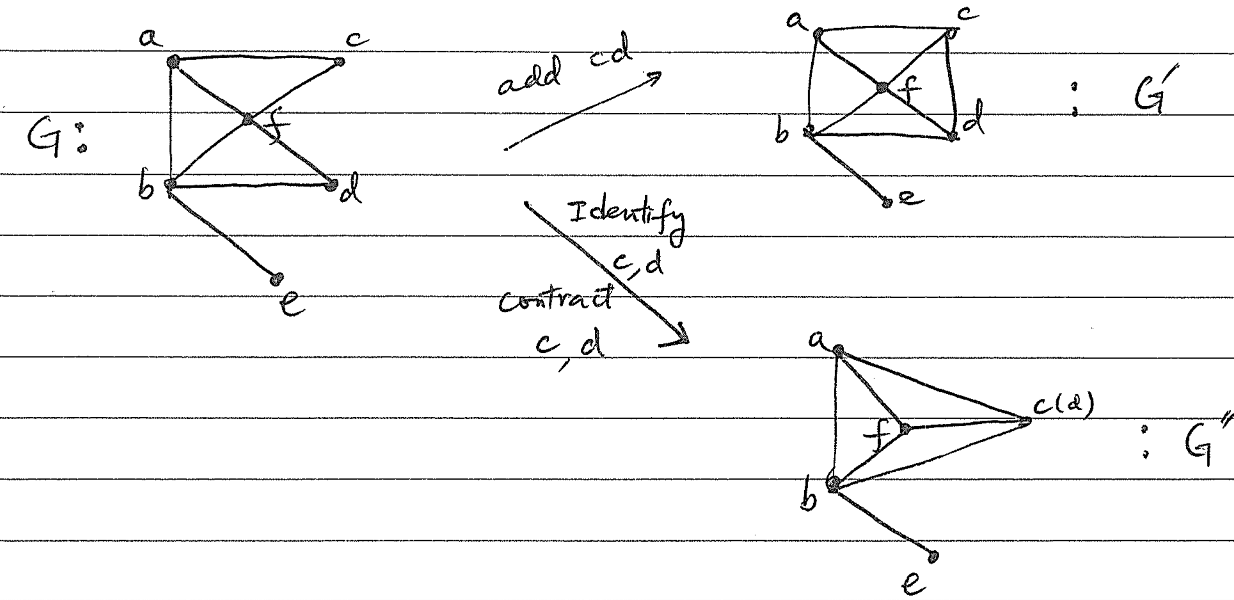


Figure 38. $\kappa(G) = 2$.

Another coloring algorithm



Observation A coloring φ of G has two outcomes:

(1) $\varphi(c) \neq \varphi(d)$ and (2) $\varphi(c) = \varphi(d)$.

Theorem 59

Let $p_H(k)$ be the number of distinct k-colorings of H. Then, $p_G(k) = p_{G'}(k) + p_{G''}(k)$ where G' and G'' are graphs obtained from G by adding xy and contracting x and y respectively for $x \not\sim_G y$.

($p_G(k)$ is known as the chromatic polynomial of G with k-colorings.)

Proof. The proof follows by the fact that $\varphi(x) = \varphi(y)$ or $\varphi(x) \neq \varphi(y)$ but not both. ∎

(∘) G is k-colorable if and only if $p_G(k) \geq 1$.

(∘) $\chi(G) = \min\{\chi(G'), \chi(G'')\}$.

Theorem 60

Let G be a (p, q)-graph with k components. Then,

$p_G(x) = \sum_{i=0}^{p-k} (-1)^i \cdot a_i \cdot x^{p-i}$, where $a_0 = 1$, $a_1 = q$ and a_i is a positive integer for $0 \leq i \leq p-k$.

Proof. By induction on $p+q$. Clearly, it's true for $p+q=1$. Assume the assertion is true for the cases of smaller $p+q$ and let G be a (p,q)-graph with k components. First, if $m=0$, then $p=k$, so $p_G(x) = x^p$, thus $a_0 = 1$, $a_1 = q = 0$. Now, consider $m \geq 1$. Let uv be an edge of G and $G_0 = G - uv$. By induction,

$p_{G_0}(x) = x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i \, x^{p-i}$ where b_i is a non-negative integer for each i. (G_0 has at least k components) Also,

$p_{G_0''}(x) = x^{p-1} - \sum_{i=2}^{p-k} (-1)^i c_i \, x^{p-i}$ where c_i is a positive integer for each i.

Note that $G_0' \cong G$ (adding uv back).

$$P_G(x) = P_{G_0}(x) - P_{G_0''}(x)$$

$$= x^p - (q-1)x^{p-1} + \sum_{i=2}^{p-k} (-1)^i b_i x^{p-i}$$

$$- x^{p-1} + \sum_{i=2}^{p-k} (-1)^i c_i x^{p-i}$$

$$= x^p - q \cdot x^{p-1} + \sum_{i=2}^{p-k} (-1)^i (b_i + c_i) x^{p-i}$$

$$= x^p - q x^{p-1} + \sum_{i=2}^{p-k} (-1)^i a_i x^{p-i}, \quad a_i > 0 \text{ for each } i. \quad \blacksquare$$

(c) If T is a tree of order p, then for each $k \geq 1$, there are $k \cdot (k-1)^{p-1}$ different k-colorings of T, i.e., $P_T(k) = k \cdot (k-1)^{p-1}$.

$$\| $$

$$k^p - \binom{p-1}{1} \cdot k^{p-1} + \cdots = k^p - (p-1)k^{p-1} + \cdots .$$