(•) The following theorem considers pseudographs, i.e., loops and multiedges are allowed.

Theorem 51 (Euler-Poincaré Theorem)

Let G be a (p,q)-pseudograph which has a 2-cell embedding on S_n. Then, $p-q+f = 2-2n$ where f is the number of faces in the embedding.

Proof. By induction on n and it's true when $n=0$ (by Euler's planar graph formula). Assume that the assertion is true when $n=k\geq 0$ and G is a (p,q)-pseudograph which has a 2-cell embedding on S_{k+1}. Since $k+1\geq 1$, there exists a handle in the embedding, see Figure 34(a). It suffices to consider the embedding such that there exists at least one edge which passes through the handle (on the surface). Note that if we can pull back an edge without passing the handle, then pull it back, see Figure 34(b). Now, we apply the idea of "cut and past" to obtain a 2-cell embedding of $\tilde{G}$ on S_k.

By using a circle around the handle, we can cut the handle through the circle and obtain $\tilde{G}$, see Figure 34(c). As a consequence the graph $\tilde{G}$ is embedded in S_k. If there are t edges passing through the handle. Then $|\tilde{G}| = p+2t$, $\|\tilde{G}\| = q+3t$ and the embedding in S_k has $f+t+2$ faces. Hence,

$$(p+2t)-(q+3t)+(f+t+2) = 2-2k.$$

This implies that $p-q+f = 2-2(k+1)$. ∎

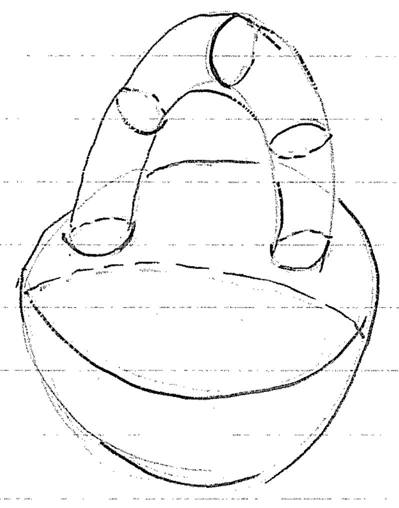


Figure 34 (a)

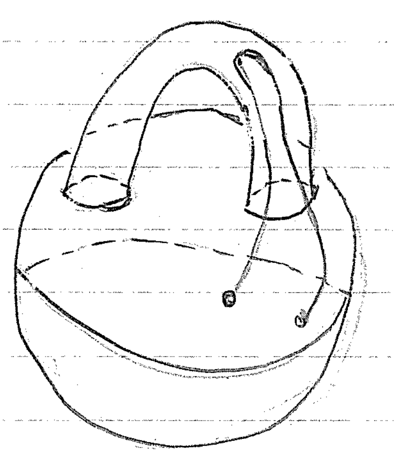


Figure 34 (a), pull edges back

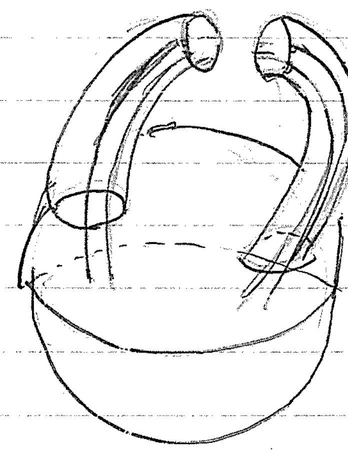
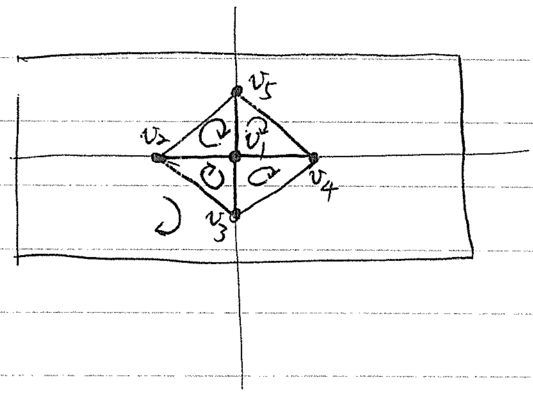


Figure 34 (c)

<u>Exercise</u> Give a more precise proof.

(∘) How to find a 2-cell embedding on S_k?



The above figure provides an embedding of K_5 on S_1, their regions are $((v_1, v_2, v_5))$, $((v_1, v_3, v_2))$, $((v_1, v_4, v_3))$, $((v_1, v_5, v_4))$, and

$((v_2, v_3, v_5, v_2, v_4, v_5, v_3, v_4))$.

(∘) Each arc of D_5 occurs exactly once in a region (face).

(∘∘) By considering each vertex, we observe that there are five permutations to determine this embedding.

$\pi_1 = (2, 3, 4, 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$ (Defined on v_1.)

$\pi_2 = (3, 1, 5, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 3 & 4 \end{pmatrix}$ (Defined on v_2)

$\pi_3 = (4, 1, 2, 5)$

$\pi_4 = (3, 2, 5, 1)$

$\pi_5 = (1, 4, 3, 2)$

On the other hand, if we are given five permutations one from each vertex, then we have a 2-cell embedding.

For example, $\pi_1 = (3,2,4,5)$, $\pi_2 = (3,1,5,4)$, $\pi_3 = (4,1,2,5)$, $\pi_4 = (3,2,5,1)$ and $\pi_5 = (1,4,3,2)$.

Now, if we start from the arc (v_1, v_2), then the permutation of π_2 will give the next vertex of the oriented 2-cell containing $(1,2)$.

$v_1 - v_2 - v_{\pi_2(1)} \Rightarrow v_1 - v_2 - v_5 - v_{\pi_5(2)} \Rightarrow v_1 - v_2 - v_5 - v_1 - v_{\pi_1(5)}$

$\Rightarrow v_1 - v_2 - v_5 - v_1 - v_3 \Rightarrow \cdots \Rightarrow ((v_1, v_2, v_5, v_1, v_3, v_2, v_1, v_4, v_3))$.

$(v_{\pi_3(4)} = v_1$ and $v_{\pi_1(3)} = v_2)$

(*) Therefore, given a (p,q)-graph G, we can define p permutations for p vertices v_i such that each permutation is a cycle using the indices of vertices in $N_G(v_i)$. Then, a 2-cell embedding will be obtained.

<u>Theorem 52</u> (The (Heffter) Rotational Embedding Scheme)

Let G be a nontrivial connected graph with $V(G) = \{v_1, v_2, \cdots, v_p\}$. For each 2-cell embedding of G on a surface, there exists a unique p-tuple $(\pi_1, \pi_2, \cdots, \pi_p)$ where for $i = 1, 2, \cdots, p$, π_i

is a cyclic permutation of $V(i)$ that describes the subscripts of the vertices in $N_G(v_i)$ in counterclockwise order about v_i. Conversely, for each p-tuple $(\pi_1, \pi_2, \cdots, \pi_p)$, there exists a 2-cell embedding of G on some surface such that for $i = 1, 2, \cdots, p$, the subscripts of the vertices adjacent to v_i are in counterclockwise order about v_i are given by π_i. Moreover, the set $\{\pi_1, \pi_2, \cdots, \pi_p\}$ induces a mapping π such that $\pi((v_i, v_j)) = \pi(v_i, v_j) = (v_j, v_{\pi_j(i)})$ for each pair of adjacent vertices v_i and v_j, $1 \le i \ne j \le p$.

proof. The scheme was first observed and used by Dyck (1888) and Heffter (1891). A formalized version was obtained later in 1960 by Edmonds.

The main idea has been mentioned before the statement of Theorem 52. The genus of surface can be obtained after the number of faces (regions) has been determined. $\square$

<u>Theorem 53</u> $\gamma(K_{2m,2n}) = (m-1)(n-1)$, $m \le n$.

Proof. For convenience, let $K_{2m,2n} = (A_o, A_e)$ where $A_o = \{a_1, a_3, \cdots, a_{4m-1}\}$ and $A_e = \{a_2, a_4, \cdots, a_{4n}\}$. Note that m may not be equal to n. See Figure 35 for the case $2m = 6$ and $2n = 8$.

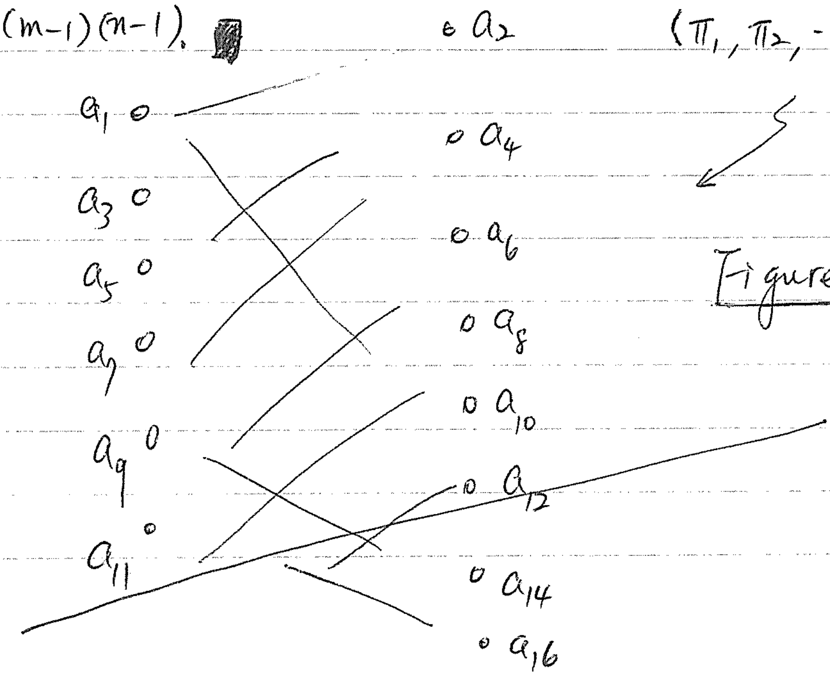
Now, let

$$\begin{cases} \pi_1 = \pi_5 = \cdots = \pi_{4m-3} = (2\ 4\ 6\ \cdots\ 4n); \\ \pi_3 = \pi_7 = \cdots = \pi_{4m-1} = (4n\ 4n{-}2\ \cdots\ 6\ 4\ 2); \\ \pi_2 = \pi_6 = \cdots = \pi_{4n-2} = (1\ 3\ 5\ \cdots\ 4m{-}1); \text{ and} \\ \pi_4 = \pi_8 = \cdots = \pi_{4n} = (4m{-}1\ 4m{-}3\ \cdots\ 5\ 3\ 1). \end{cases}$$

We may check that this embedding does have $2mn$ regions each of them is bounded by a 4-cycle. Hence, by Theorem 51,

$$2\gamma(G) = 2 - p + q - r = 2 - (2m+2n) + 4mn - 2mn = 2mn - 2m - 2n + 2.$$

Hence, $\gamma(G) = (m-1)(n-1)$. ∎



$(\pi_1, \pi_2, \cdots, \pi_{12}, \pi_{14}, \pi_{16})$

<u>Figure 35</u>, $K_{6,8}$

(∘∘) There are $\prod_{i=1}^{p}(\deg_G(v_i)-1)!$ p-tuples of $(\pi_1, \pi_2, \cdots, \pi_p)$.

(∘∘∘) The 2-cell embedding with the largest number of faces gives the genus of G.

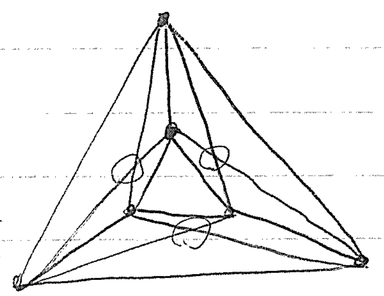
(*) The 2-cell embedding with the smallest number of faces gives the "maximum" genus of G, denoted by $\gamma_M(G)$.

(**) Finding $\gamma(G)$ is a very difficult problem in general.

(**) Finding $\gamma_M(G)$ is comparatively easier.

<u>Theorem 54</u>. Let $cr(G)$ denote the crossing number of G. Then, $cr(K_5) = cr(K_{3,3}) = 1$ and $cr(K_6) = 3$.

<u>Proof</u>. Since $\gamma(K_5) = \gamma(K_{3,3}) = 1$, the proof follows by a drawing with "1" crossing number. Now, we consider K_6. By Figure 35, $cr(K_6) \le 3$. Let $cr(K_6) = k$. Then, we may convert the crossings into new vertices. Hence, we have a planar graph G (obtained from above): $|G| = 6 + k$ and $\|G\| = 15 + 2k$. By the fact $15 + 2k \le 3(6+k) - 6$, we have $k \ge 3$. ∎



<u>Figure 35</u>

Theorem 55 $cr(K_9) = 36$.

Proof. For the upper bound, it suffices to give a drawing which has exactly 36 crossings. But, it is very technical to show the lower bound. Here, we provide a drawing for $cr(K_9) \leq 36$, see Figure 36.

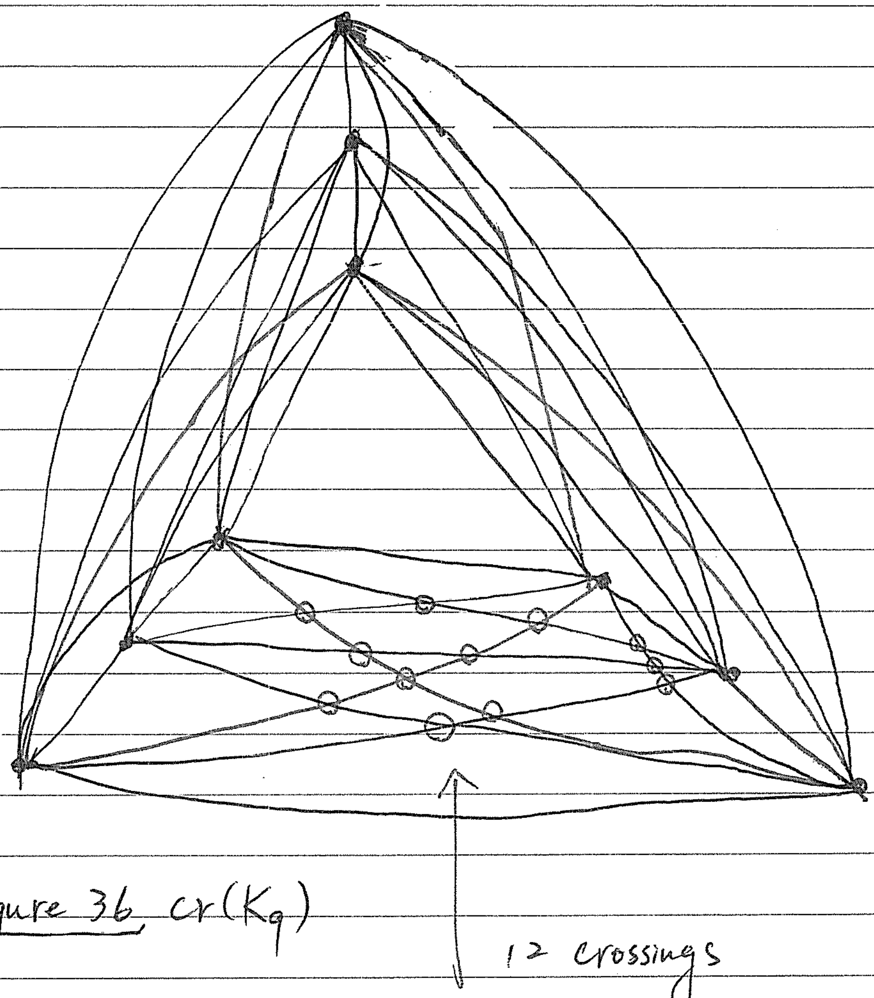


Figure 36, $cr(K_9)$

Conjecture $cr(K_p) = \frac{1}{4}\left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor\left\lfloor\frac{p-3}{2}\right\rfloor$.

(True for $1 \leq p \leq 10$.)