Theorem 41 $R(p_1, p_2, \cdots, p_t) \leq R(p_1-1, p_2, \cdots, p_t) + R(p_1, p_2-1, p_3, \cdots, p_t) + \cdots + R(p_1, p_2, \cdots, p_t - 1) - t + 2$.

Proof. By a similar argument as the proof $R(s,t) \leq R(s-1,t) + R(s,t-1) - 1$. ∎

(i) $R(3,3,3) \leq 6+6+6-3+2 = 17$ (Theorem 41).

(ii) There exists a 3-edge-coloring of K_{16} such that no mono-chromatic triangles occur.

Theorem 4.2 $R(\underbrace{3,3,\cdots,3}_{k\text{-tuples}}) \underset{def}{=} R_k(3) \leq \lfloor e \cdot k! \rfloor + 1$.

Proof. Since $R(3,3) = 6$, $R(3,3,3) = 17$, the assertion is true for $k = 2$ and 3. Assume that it holds for $k-1$ when $k > 3$. Hence, $R_{k-1}(3) \leq \lfloor e(k-1)! \rfloor + 1$. By Theorem 41,

$$R_k(3) \leq k(\lfloor e(k-1)! \rfloor + 1) - k + 2$$

$$= k\lfloor e(k-1)! \rfloor + 2.$$

Now, $k\lfloor e(k-1)! \rfloor = k\lfloor (k-1)! \cdot (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(k-1)!} + \frac{1}{k!} + \cdots) \rfloor -$

$$= k\lfloor M + \frac{1}{k} + \frac{1}{(k+1)k} + \frac{1}{(k+2)(k+1)k} + \cdots \rfloor = \cdots$$

$(?) = \lfloor (k!) \cdot e \rfloor - 1$. ∎

(0) Instead of $R(s,t)$, we use $R(H_1, H_2)$ to denote the smallest integer n such that any 2-edge-coloring (red, blue) of K_n, either there exists a red H_1 or a blue H_2.

<u>Theorem 43</u> $R(C_4, C_4) =_{def} R(C_4) \leq 8$.

<u>Proof.</u> Consider a graph $G \not\supseteq C_4$ and $|G| = 8$. By Theorem 35-1, $\|G\| \leq \frac{1}{2} z(8,8;2,2) \leq \frac{n\cdot(1+\sqrt{4n-3})}{4} = 2(1+\sqrt{29}) < 14$. That is, if a graph G is of order 8 and size 14, then $G \supseteq C_4$. Now, in a 2-edge-colored K_8, either red or blue edges induce such a graph, the proof follows. ∎

<u>Theorem 44</u> $R_k(C_4) \leq k^2+k+2$.

<u>Proof.</u> Let $n = k^2+k+2$ and consider a k-edge-colored K_n. Since $ex(n; C_4) \leq \frac{n}{4}(1+\sqrt{4n-3}) = \frac{k^2+k+2}{4}\cdot(1+\sqrt{4k^2+4k+5})$

$\leq \frac{k^2+k+2}{2} \cdot \frac{1+\sqrt{4k^2+4k+5}}{2}$ ....

Now, compare (k^2+k+1) and $\frac{k(1+\sqrt{4k^2+4k+5})}{2}$. By direct calculatio[n] we have $k^2+k+1 > \frac{k(1+\sqrt{4k^2+4k+5})}{2}$. This implies that

$k\cdot ex(n; C_4) \leq \frac{k^2+k+2}{2} \cdot \frac{k(1+\sqrt{4k^2+4k+5})}{2} < \frac{(k^2+k+2)(k^2+k+1)}{2} = \binom{n}{2}$. ∎

(*) $R(H_1, H_2) = \min.\{n \mid |G| = n,\ G \geq H_1 \text{ or } \overline{G} \geq H_2\}$.

Theorem 45

For $l \geq 1$ and $p \geq 2$, $R(lK_2, K_p) = 2l + p - 2$.

Proof. Let O_k denote the graph of order k which contains no edges.
(Stable set)
Let H be a graph of order $2l+p-3$ such that $H = O_{p-2} \cup K_{2l-1}$ (see Figure 30).

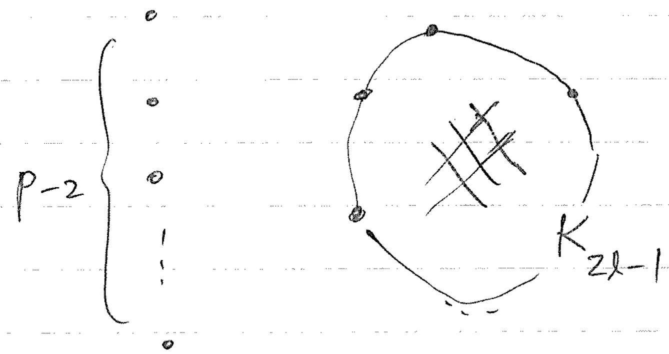

Figure 30

So, it is not difficult to see that H does not contain a matching of size l. As a consequence, $\overline{H} = O_{2l-1} \vee K_{p-2}$ (join) does not contain a K_p. Hence, $R(lK_2, K_p) \geq 2l + p - 2$.

On the other direction of inequality, let $n = 2l + p - 2$ and assume that a graph G of order n does not contain a matching of size l. Let $s < l$ be the maximum number of independent edges in G.

Now, consider $\overline{G}$. Since in G, the set of vertices not incident to these s edges induces a graph of order $n-s$ which has no edges, $\overline{G}$ contains a complete graph of order $n-2s \geq n-2(l-1) = p$. This concludes the proof. ■

Appendix

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Use max-flow min-cut theorem to prove Hall's Theorem.

Proof. Let $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, $m \le n$.

Claim: There exists a matching in $G = (X, Y)$ saturates X if and only if for each subset $A \subseteq X$, $|\Gamma(A)| \ge |A|$.

(Note) We shall prove the theorem for both $(\Rightarrow)$ and $(\Leftarrow)$ by using network argument.

First, we construct a network N by (1) adding s and t such that (s, x_i) and (y_j, t) are arcs in N, $1 \le i \le m$ and $1 \le j \le n$; (2) orienting $x_i y_j \in E(G)$ with (x_i, y_j), and (3) $c(s, x_i) = c(y_j, t) = 1$ and $c(x_i, y_j) = M > |X| = m$. (See Figure 1 below.)

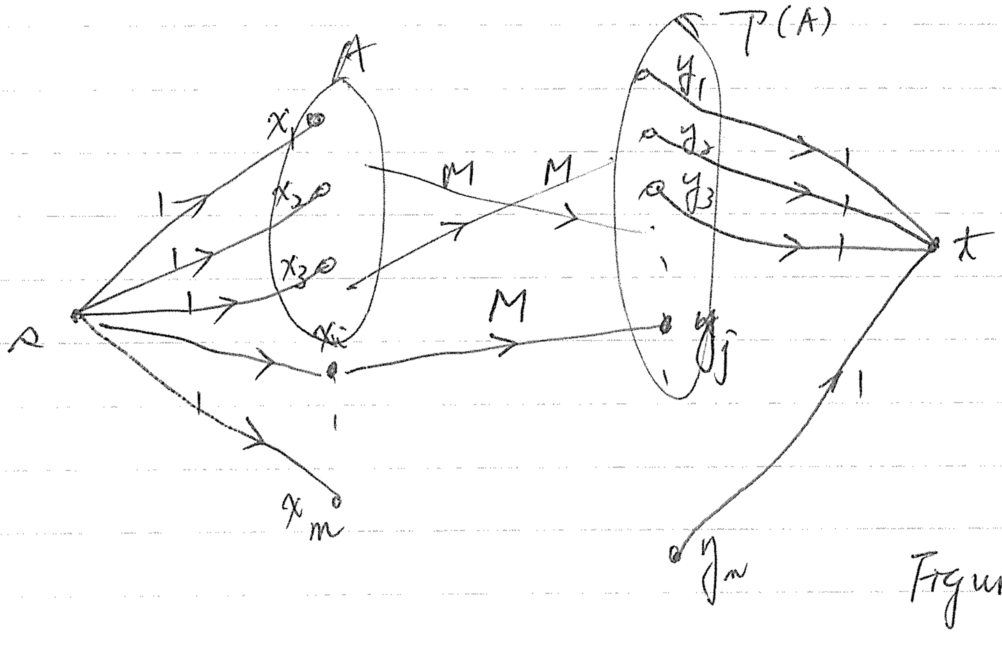

Figure 1.

Now, it is easy to see that a flow of value $|X| = m$ will provide a matching saturates X since each (y_j, t) can take at most one flow value through the flow.

$(\Rightarrow)$ Suppose that there exists a subset A of X such that $|A| > |\Gamma(A)|$. Let $S = \{s\} \cup A \cup \Gamma(A)$. Then, $\langle S, \bar{S} \rangle$ is a cut with capacity $\underbrace{(|X|-|A|)}_{(s \to \bar{S})} + \underbrace{|\Gamma(A)|}_{(S \to t)}$ since there are no arcs from the vertices of A to the vertices of $Y \setminus \Gamma(A)$. This capacity of cut is less than $|X|$, Hence, there exists no flow with value $|X|$ and thus no matching saturates X.

$(\Leftarrow)$ Assume that $|\Gamma(A)| \geq |A|$ for each $A \subseteq X$. It suffices to claim that all cuts have capacity at least $|X|$. Let $S = \{s\} \cup A \cup B$, see Figure 2. ($\underbrace{A}_{} \subseteq X$ and $\underbrace{B}_{} \subseteq Y$.) They can be empty.

(B is not necessarily be $\Gamma(A)$.)

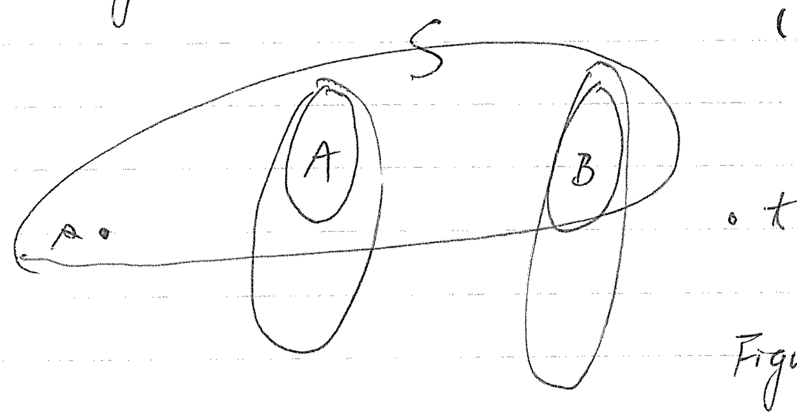

Figure 2.

Now, if there exists an arc from the vertices of A into $Y \setminus B$, then $c(S,\bar{S}) \geq M > |X|$. On the other hand, if $\underline{\Gamma(A) \subseteq B}$, then $c(S,\bar{S}) = \underbrace{(|X|-|A|)}_{s \to \bar{S}} + |B| \geq |X| - |A| + |\Gamma(A)| \geq |X|$. Since $c(S,\bar{S}) = |X|$ in the case $S = \{s\}$, we obtain a min-cut with capacity $|X|$ and thus there exists a flow with maximum value $|X|$. The proof follows. $\blacksquare$

Use max-flow min-cut theorem to prove Menger's Theorem.

Proof. First, we prove a directed version of Menger's Theorem.

(i) If s and t are distinct vertices of a digraph D such that $s \not\to_D t$, then the maximum number of internally disjoint u–v directed paths in D equals the minimum number of vertices in a u–v separating set in D.

(ii) For the undirected version, we replace each edge uv by a pair of arcs (u,v) and (v,u).

Proof of (i).

Let $\tilde{D}$ be the digraph obtained as follows:

(1) $\forall\ x \in V(D) \setminus \{s,t\}$, split x into two vertices x' and x'', also let $(x', x'') \in A(\tilde{D})$;

(2) $\forall\ (x,y) \in A(D)$ such that $\{x,y\} \cap \{s,t\} = \phi$, replace (x,y) with (x'', y');

(3) For $x \neq t$, $(s,x) \in A(D)$, $(x,s) \in A(D)$, replace them with (s,x') and (x'',s) respectively, and

(4) Replace (t,x) with (t,x') and (x,t) with (x'',t) if $x \neq s$.

As a consequence, we have a network N defined on D by assigning each arc a capacity "1".

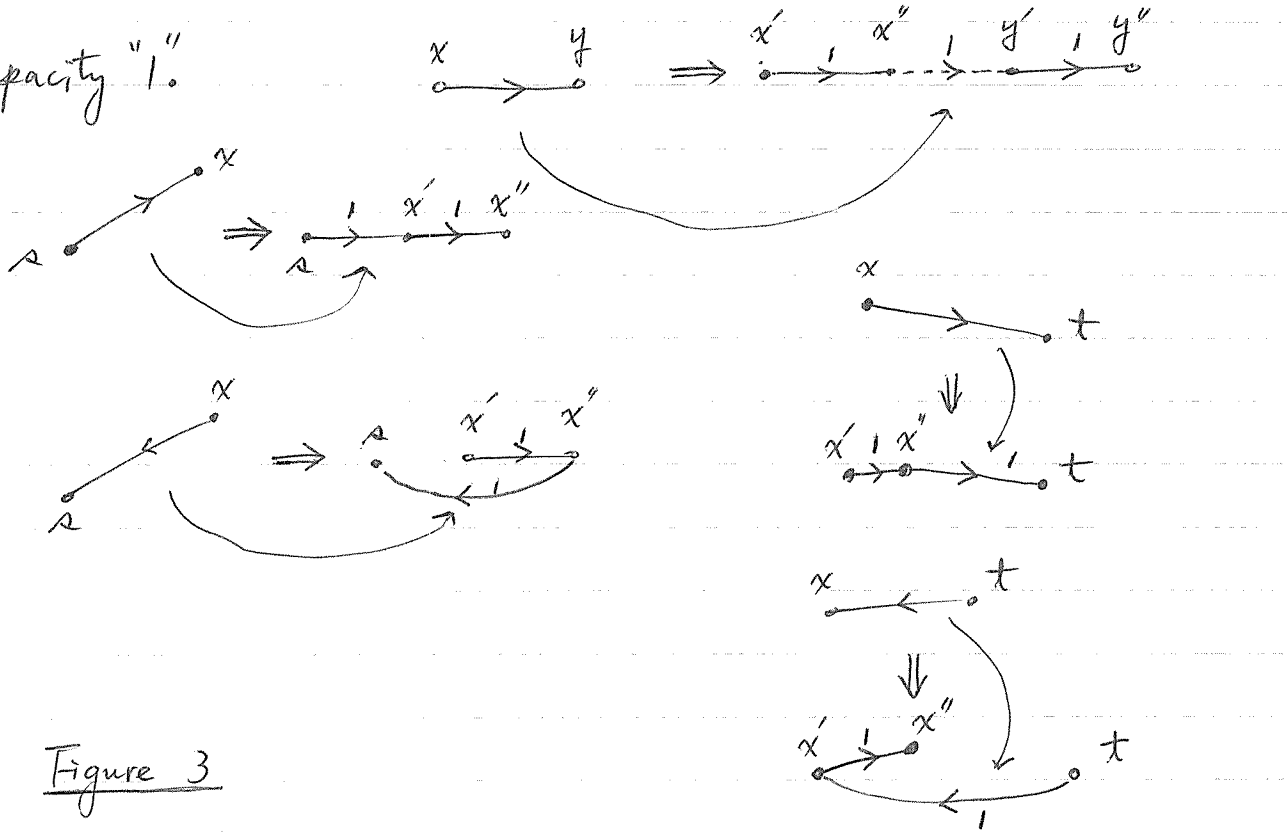

Figure 3

Let m be the maximum number of internally disjoint s-t paths in D and n be the minimum number of vertices in an s-t separating set in D. Let A be a u-v separating set of arcs in $\tilde{D}$ and $|A| = n'$.
First, we observe that if $(S, \bar{S})$ is a cut, then $cap(S, \bar{S}) \le n'$.
Here, S contains s if $(s, x') \in A$, S contains x if $(x', x'') \in A$ and $(x'', t) \in A$.
Moreover, $cap(S, \bar{S}) \ge n$. As a matter of fact, the min-cut is of capacity n and thus, N has a maximum flow n. By the way,

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the construction of network shows that a flow value 1 will give a path from s to t. Since each arc has capacity one, these (directed) paths are internally disjoint. The proof follows. ∎