Theorem (Euler)

Let G be a connected planar graph with p vertices, q edges and f faces (regions). Then $p-q+f=2$.

Proof. By induction on q. Since G is connected, G has at least $p-1$ edges. (?) If G has $p-1$ edges and G is connected, then G is a tree which contains no cycles. This implies that $f=1$ and thus $p-(p-1)+1=2$. The assertion is true for "minimal" graphs. Let the hypothesis is true for $\|G\| = k \geq p-1$. Now, consider G with $k+1$ edges. Clearly, G contains a cycle. Let e be a cycle edge. Since G is a connected planar graph (with q faces), $G-e$ is also a connected planar graph. Moreover, $\|G-e\| = k$ and $G-e$ has k edges and $q-1$ faces. By induction $p-k+q-1=2$ and thus

$p-(k+1)+q=2$. This concludes the proof. ∎

with largest size

<u>Theorem</u> If G is a planar graph with largest size, then $\|G\| = 3|G| - 6$.

<u>Proof.</u> By observation, if G has maximum size, then each region of G is a Triangle. Since each edge of G is in the boundary of exact two regions, $3 \cdot f = 2 \cdot q$ where f is the number of regions and q is the size of G, i.e., $q = \|G\|$. Now, by Euler's formula $p - q + f = 2$ equivalently

$$|G| - \|G\| + \frac{2}{3}\|G\| = 2$$

$$\Rightarrow 3 \cdot |G| - 6 = \|G\|. \quad (G \text{ is a maximal planar graph!})$$ ∎

<u>Corollary</u> If G is a planar graph, Then $\|G\| \leq 3 \cdot |G| - 6$.

<u>Corollary</u> In any planar graph, there exists at least one vertex of degree smaller than 6.

<u>Corollary</u> The degree sum of a planar graph is at most $6 \cdot |G| - 12$.

This corollary is very useful.

Theorem (5-color Theorem)

If G is a planar graph, then $\chi(G) \leq 5$.

Proof. It suffices to prove the theorem for maximal planar graph. By induction on $|G|$ and it is true for graphs of smaller orders. Assume the assertion is true for all graphs (planar) of order k and let G be a maximal planar graph of order $k+1$.

Since G is maximally planar, $\delta(G) \geq 3$. By the collary mentioned above, we have three cases to consider: $\delta(G) = 3$ or 4 or 5.

Case 1. $\delta(G) = 3$

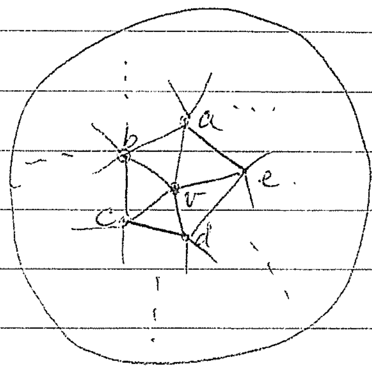
Case 2. $\delta(G) = 4$

Easy to see. (?)

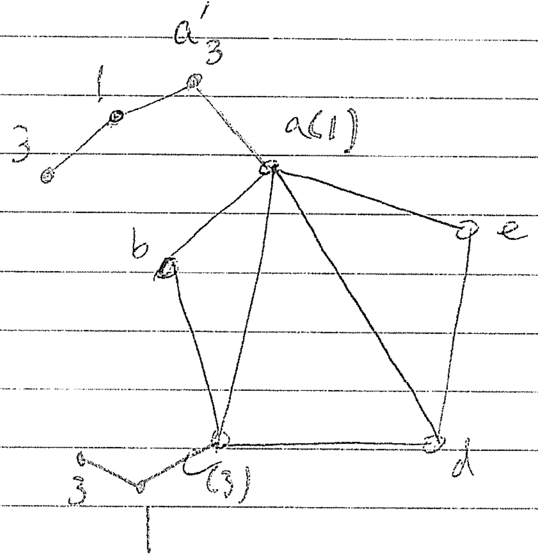
Case 3. $\delta(G) = 5$

Let $\deg(v) = 5$.

Consider $\tilde{G} = (G - v) + ac + ad$.



By induction hypothesis, $\chi(\tilde{G}) \leq 5$. Now, if the set of vertices $\{a,b,c,d,e\}$ uses at most 4 colors, then the vertex v in $V(G)$ can be colored by a distinct color, and we have the proof. On the other hand, if $\{a,b,c,d,e\}$ uses five colors, say $\varphi(a)=1$, $\varphi(b)=2$, $\varphi(c)=3$, $\varphi(d)=4$ and $\varphi(e)=5$, then we need recolor $\tilde{G}$ as follows.



First, we consider the chains starting from a and c respectively in which the vertices are colored 1 and 3. If one of them terminate somewhere different from the other vertex (a or c), say $\langle a, \cdots \rangle$, then recolor a by 3 and a' by 1, ⋯, etc.

Therefore, the colors used are 4 and we have a color for v. On the other hand, if those two chains are the same, then instead of changing the colors, we consider the chains starting at b and d respectively (colored 2, 4, 2, ...). Since $\tilde{G}$ is a planar graph, these two chains can not be the same, and thus we can recolor one of them, say from 2, 4, 2, ... to 4, 2, 4, ... Again, we have an extra color for v. This concludes the proof. ▨

(*) The fact that in a planar graph G, $\delta(G) \leq 5$ is very important.

Definition (d-degenerate)

A graph G is called d-degenerate if for each subgraph H of G, $\delta(H) \leq d$.

Therefore, a tree is 1-degenerate and a planar graph is 5-degenerate. Clearly, an r-regular graph is r-degenerate.

Theorem (about vertex-coloring)

If G is d-degenerate, then $\chi(G) \leq d+1$.

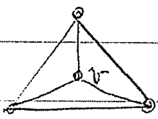
Proof. Suppose not. Let G be a counter-example which has minimum order. That is, for each $v \in V(G)$, $\chi(G-v) \leq d+1$, but $\chi(G) = d+2$. Now, let $\deg(v) = \delta(G)$ for some $v \in V(G)$. Since $\chi(G-v) = d+1$ and $\deg(v) \leq d+1$, $\chi(G) \leq d+1$. $\rightarrow\leftarrow$ ∎

We can also use (*) to prove the well-known Wagner's theorem.

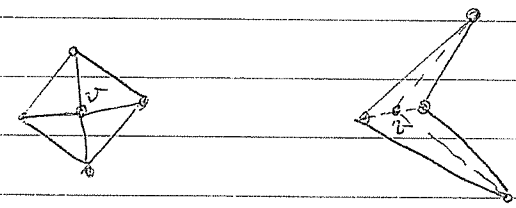
<u>Theorem.</u> If G is a planar graph, then G has a drawing on a plane such that each edge is a straight line segment.

<u>Proof.</u> It suffices to consider the case when G is a maximal planar graph (the boundary of each region is a triangle). Suppose that G is such a graph of smallest order in which no straight segment drawing is possible. That is, for each $v \in V(G)$, $G-v$ is a planar graph and $G-v$ has a straight segment drawing. Now, let $\deg(v) = \delta(G) \le 5$. Then, the drawing of G is be obtained by considering the following cases.

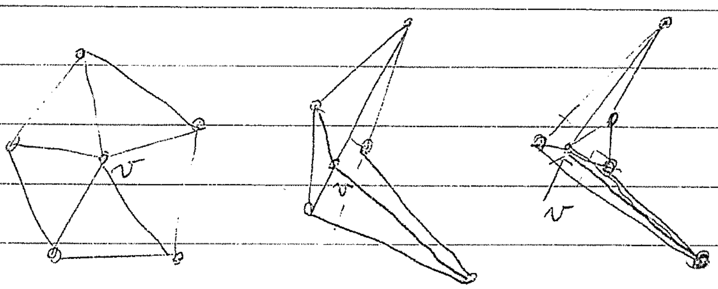
(1) $\deg(v) = 3$.



(2) $\deg(v) = 4$.



(3) $\deg(v) = 5$.



Theorem (Regular Polyhedra)

There are exactly five regular polyhedra.

Proof. Clearly, a regular polyhedron is a planar graph, moreover, it is a regular graph. Since they are polyhedra, the corresponding graphs are either 3-regular, 4-regular or 5-regular. Hence, we have the following equalities:

$$\begin{cases} (1) & p - q + r = 2 \quad (k\text{-regular and the boundary of each face is of length } l) \\ (2) & kp = 2q = rl \end{cases}$$

$$\Rightarrow q\left(\frac{2}{k} - 1 + \frac{2}{l}\right) = 2 \text{ and } \frac{2}{k} - 1 + \frac{2}{l} > 0$$

$$\Rightarrow (k-2)(l-2) < 4. \text{ Since } k, l \geq 3,\ k, l \leq 5.$$

This implies that there are at most 9 possibilities for (k, l). But, $(4,4), (4,5), (5,4), (5,5)$ are not possible. This implies that we have at most five regular polyhedra. Now, by the existence of tetrahedron, cube, octahedron, dodecahedron and icosahedron, we conclude the proof. ∎

Graphs which are not planar

<u>Fact</u> If the girth of a graph G is $g(G)=g$, and G is planar, then $\|G\| \le \frac{g}{g-2}(p-2)$.

<u>proof.</u> Since $g \cdot r \le 2q$ and $p-q+r=2$, we conclude that $p-2 \ge q \cdot \frac{g-2}{g}$.

<u>Fact</u> $K_{3,3}$ is not a planar graph.

The materials about topological graph theory are attached for reference in next pages. (the following)

<u>Theorem</u> (Euler–Poincaré)

Let G be a (p,q)-graph which has a 2-cell embedding in an orientable suface of genus n. Then $p-q+r = 2-2n$ where r is the number of regions.

<u>Note</u> K_5 can be drawn on S_1 with 5 regions.
genus 1

Crossing Number

Definition (Crossing number)

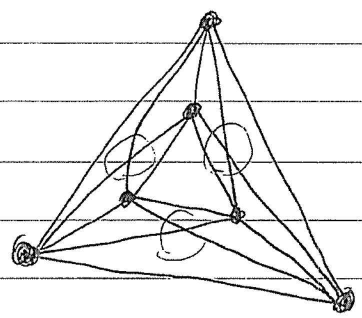
The crossing number of a graph G, denoted $cr(G)$, is the minimum number of crossings among all drawings of G on a plane (or a sphere).

(tangling)
(maximum)

Fact If G is not a planar graph, then $cr(G) \geq 1$.

Fact A drawing of G on a plane with crossing number k gives an upper bound of $cr(G)$.

Example A drawing of K_6 with "3" crossings

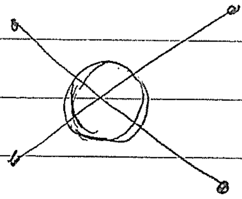


Fact $cr(K_6) = 3$

Proof. Note that if we change the crossings into vertices, then we have a planar graph.

Let $cr(K_6) = k$. Then $\tilde{K}_6$ (the graph from K_6 after assigning the crossings as vertices) has $6+k$ vertices, $15+k$ edges. Since $\tilde{K}_6$ is a planar graph $15+2k \geq 3(6+k)-6$, hence $k \leq 3$. ■

(A crossing gives "2" more edges !)



?

$$\begin{cases} \frac{1}{64} n(n-2)^2(n-4), & n \text{ even} \\ \frac{1}{64}(n-1)^2(n-3)^2, & n \text{ odd.} \end{cases}$$

Fact $cr(K_5) = 1$, $cr(K_{3,3}) = 1$

Problems Determine $cr(K_n)$ and $cr(K_{m,n})$.

(Research problems) $\overset{?}{=} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$

Conjecture If $\chi(G) = k$, then $cr(G) \geq cr(K_k)$.

The conjecture holds for $k \leq 4$ (trivial !?)

Remark

It is not easy to determine both crossing number and tangling number of a graph.

7-12

Chapter 5 Topological Graph Theory

§ 5.1. Basic Notations

Topological graph theory studies the "drawing" of a graph on a surface. A proper drawing on a surface of a graph G with $|G| = p$ and $||G|| = q$ follows the rules :

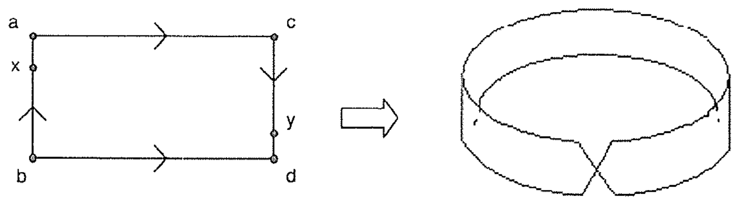
(1). There are p points on the surface which corresponds to the set of vertices in G; and

(2). There are q curves joining points defined above which correspond to the set of edges and they are pairwise disjoint except possibly for the endpoints.

- The drawing is on a surface defined on $\mathbb{R}^3$.
- A **2-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to the open unit disk defined on $\mathbb{R}^2$.
- An **n-manifold** is a connected topological space in which every point has a neighborhood homeomorphic to $B_n = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 < 1\}$.
- A subspace M of $\mathbb{R}^3$ is **bounded** if there exists a positive real number K such that $M \subseteq \{(x, y, z) \mid x^2 + y^2 + z^2 = K\}$.
- Let $M \subseteq \mathbb{R}^3$ be a 2-manifold. Then M is said to be **closed** if it is bounded and the boundary of M coincides with M.
- Let $M(\subseteq \mathbb{R}^3)$ be a 2-manifold; M is said to be **orientable** if for every simple closed C on M, a clockwise sense of rotation is preserved by traveling once around C. Otherwise, M is non-orientable.
- A 2-manifold M is orientable if and only if it is **two-sided**.

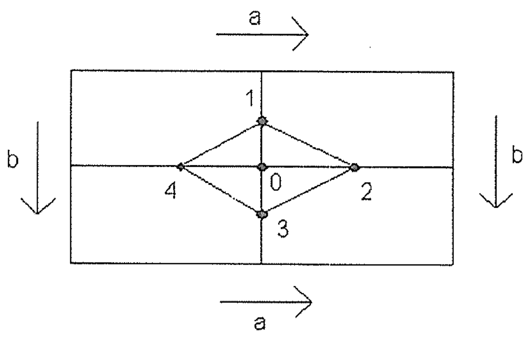
Definition 5.1.1. (Orientable Surface)
A surface is a compact orientable 2-manifold that may be thought of as a sphere on which has been placed(inserted) a number of "handles" (holes). A sphere, denoted by S_0, is the surface of a 3-dimension ball. More precisely, $S_0 = \{(x, y, z)|x^2 + y^2 + z^2 = r^2, r \in \mathbb{R}^+\}$. S_1 is known as a torus, S_2 a double torus, and S_h is a surface obtained by adding h handles to S_0.

Definition 5.1.2. (Non-Orientable Surface)
A surface obtained by adding k cross-caps to S_0 is known as the non-orientable surface N_k. (A cross cap is obtained from Möbius band described in what follows.)
Cross cap: Attach the boundary of a Möbius band to a cycle on S_0 to obtain a cross cap.

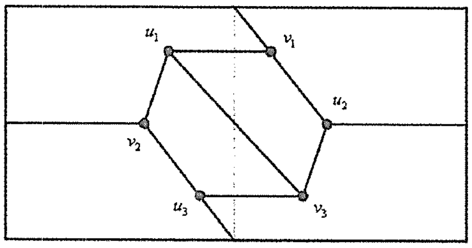


a→d, b→c, x→y

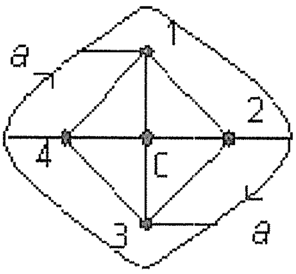
Definition 5.1.3. (Embedding or Imbedding, 2-cell embedding)
An embedding of a graph in a surface is a continuous 1-1 function from a topological representation of the graph into the **surface**. If every region of the embedding is homeomorphic to a 2-dim open disc, then the embedding is a **2-cell embedding**(圓盤嵌入).



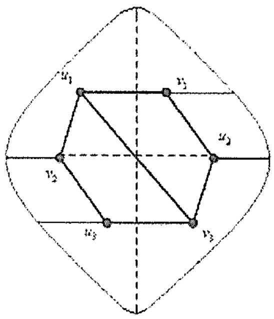
Embedding of K_5 on S_1



Embedding of $K_{3,3}$ on S_1

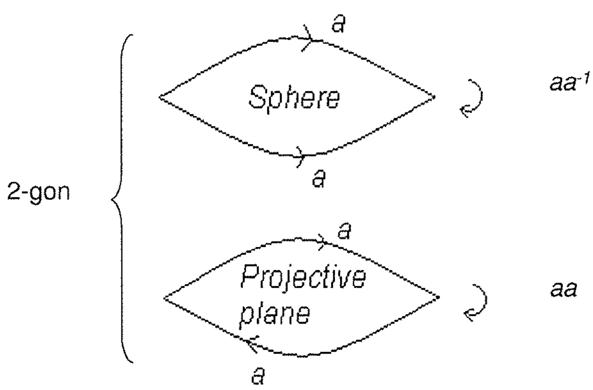


Embedding of K_5 on N_1
(N_1 is also known as a projective plane.)



Embedding of $K_{3,3}$ on N_1

- Surfaces can be represented by a polygon (standard fundamental).



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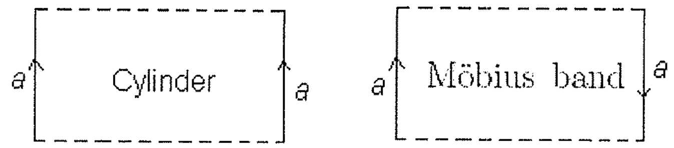
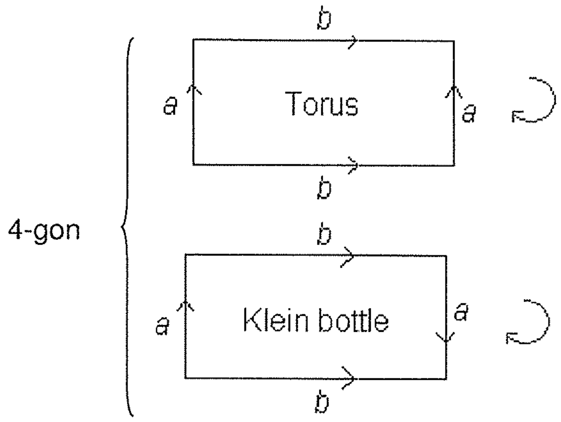
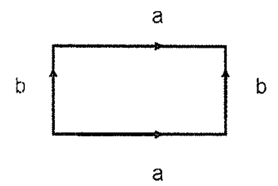




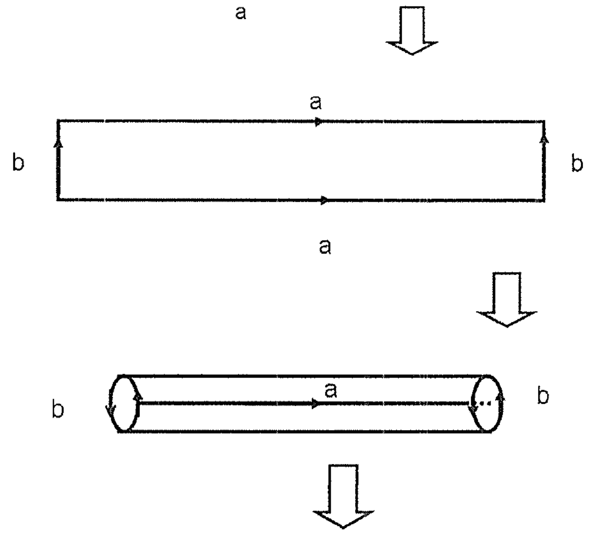
$aba^{-1}b^{-1} = a_1b_1a_1^{-1}b_1^{-1}$

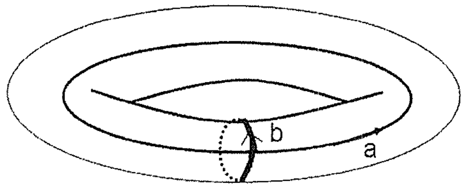
$abab^{-1} = (ab)(ab)b^{-1}b^{-1} = a_1a_1a_2a_2$

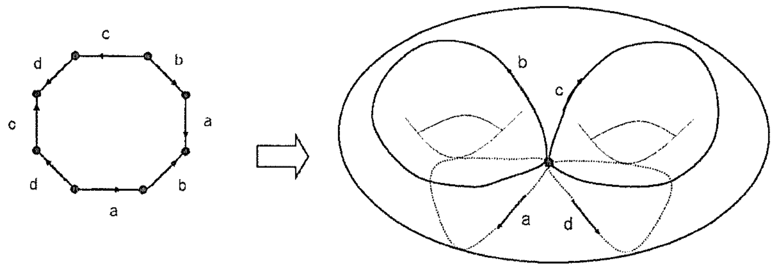
where $a_1 = ab$ *and* $a_2 = b^{-1}$.



$\boldsymbol{ab^{-1}a^{-1}b}$







(*) The standard fundamental polygon for S_k is $a_1b_1a_1^{-1}b_1^{-1}...a_kb_ka_k^{-1}b_k^{-1}$.
(**) The standard fundamental polygon for N_h is $a_1a_1a_2a_2...a_hb_h$.

Definition 5.1.4. If a graph G can be embedded in S_0, then G is a planar graph.

Theorem 5.1.1.(Euler's Formula)
Let G be a **connected planar** graph which has p vertices, q edges and r regions. Then $p - q + r = 2$.
Proof. By induction on q. □

Corollary 5.1.2.
Let G be a planar graph which has k components, p vertices, q edges and r regions. Then $p - q + r = 1 + k$.

Proof. By induction on k. $k = 1$ is true by Theorem 1. Assume the assertion is true for k. Let G be a graph with p vertices, q edges and r regions, and G have $k+1(\geq 2)$ components. Now let $\tilde{G} = G+e$ (e connects two components). Then, $\tilde{G}$ has p vertices, $q + 1$ edges, r regions and k components. Hence, $p-(q+1)+r = 1+k$. This implies $p-q+r = 1 + (k + 1)$. The proof follows. □